

BOUNDARY ACTION OF AUTOMATON GROUPS WITHOUT SINGULAR POINTS AND WANG TILINGS

DANIELE D'ANGELI, THIBAUT GODIN, INES KLIMANN, MATTHIEU PICANTIN,
AND EMANUELE RODARO

ABSTRACT. We study automaton groups without singular points, that is, points in the boundary for which the map that associates to each point its stabilizer, is not continuous. This is motivated by the problem of finding examples of infinite bireversible automaton groups with all trivial stabilizers in the boundary, raised by Grigorchuk and Savchuk. We show that, in general, the set of singular points has measure zero. Then we focus our attention on several classes of automata. We characterize those contracting automata generating groups without singular points, and apply this characterization to the Basilica group. We prove that potential examples of reversible automata generating infinite groups without singular points are necessarily bireversible. Then we provide some necessary conditions for such examples to exist, and study some dynamical properties of their Schreier graphs in the boundary. Finally we relate some of those automata with aperiodic tilings of the discrete plane via Wang tilings. This has a series of consequences from the algorithmic and dynamical points of view, and is related to a problem of Gromov regarding the searching for examples of $\text{CAT}(0)$ complexes whose fundamental groups are not hyperbolic and contain no subgroup isomorphic to \mathbb{Z}^2 .

CONTENTS

1. Introduction	2
2. Preliminaries	3
2.1. Mealy automata	3
2.2. Automaton groups	5
3. The considered classes	6
3.1. Contracting automata	6
3.2. (Bi)reversible automata	7
3.3. Automata with sink	9
4. Topological properties of the action on the boundary	10
4.1. Action on the boundary	10
4.2. Singular points	11
4.3. The contracting case	13
4.4. The bireversible case	17
5. Commuting pairs and dynamics on the boundary	18
5.1. Commuting pairs	19
5.2. The reversible-invertible case	20
6. Dynamics and Wang tilings	26
6.1. Wang tiles <i>vs</i> cross-diagrams	26
6.2. Commuting pairs on a restricted stateset	29
6.3. Aperiodic tilings and singular points	34
7. Some open problems	38
References	38

Date: April 27, 2016.

Key words and phrases. automaton groups, singular points, critical points, Schreier graphs, boundary continuity, Wang tilings, commuting pairs, helix graphs.

1. INTRODUCTION

The motivation comes from the study of the dynamical system $(G, \partial T, \mu)$ given by the measure μ preserving action of a group G on the boundary ∂T of a rooted tree T . By considering one orbit of this action (*i.e.* a Schreier graph), one may ask if it is possible to recover the information about the original dynamics in terms of the information contained in a typical orbit. This problem may be rephrased as follows: what conditions have to be imposed on the dynamical system $(G, \partial T, \mu)$ in order to guarantee that, for a typical point $\xi \in \partial T$, this dynamical system is isomorphic to the system $(G, \overline{\text{Stab}_{\langle \mathcal{M} \rangle}(\xi)}, \overline{\nu})$, for some measure $\overline{\nu}$ concentrated on the closure $\overline{\text{Stab}_{\langle \mathcal{M} \rangle}(\xi)}$? (Problem 8.2 in [15]). This problem has been studied by Y. Vorobets in the special case of the Grigorchuk group [36]. He showed that for this group it is possible to reconstruct the action of the general dynamical system on the boundary starting from the study of one orbit. His method uses the study of the map St that associates to any point in the boundary of the tree its stabilizer subgroup in the automaton group.

Motivated by these ideas, we examine the dynamical and algorithmic implications of the continuity of the map St in the context of automaton groups, and how some combinatorial properties of the generating automaton reflect into the continuity of this map. In particular, we focus our attention on several classes of Mealy automata: the contracting case (see [25]), the reversible case, the bireversible case, and finally the case of automata with a sink-state which is accessible from every state (henceforth denoted by \mathcal{S}_a). We first show that, in general, the measure of the set of the points in which St is not continuous (henceforth called singular points) is zero. In the bireversible case singular points are exactly points with non-trivial stabilizers. This reproves the well known fact that bireversible automata give rise to essentially free actions on the boundary [32, Corollary 2.10]. Driven by these facts and the question raised by Grigorchuk and Savchuk in [16] regarding the existence of singular points for the action of a bireversible automaton generating an infinite group, we generalize the previous open problem into the study of examples of automaton groups without singular points. In the case of contracting groups we provide a characterization for such automata in terms of languages recognized by Büchi automata that we call *stable automata*. In the examples that we present, we show that the situations may be very different. For instance, in contrast with the Hanoi Towers group case, we show that the Basilica group has no singular points.

In both the class of reversible invertible automata and the class \mathcal{S}_a , by using the notion of helix graph, we reduce this problem to the existence of certain pairs of words, called *commuting pairs*, that is, two words, one on the stateset the other one on the alphabet, that commute with respect to the induced actions. Using this fact we prove a series of results. For instance, it turns out that the existence of singular points is always guaranteed for the reversible invertible automata that are not bireversible. This shows that, in the class of reversible invertible automata, the core of the problem of finding examples of group automata without singular points is reduced to the class of bireversible automata. We present some necessary conditions for such examples to exist. For instance, the generated group is necessarily fully positive, that is, it is defined by relators that do not contain negative occurrences of the generators. Furthermore, we prove that if a bireversible automaton generates an infinite non-torsion group, then having all stabilizers in the boundary that are

torsion groups (like in the situation of not having singular elements) is equivalent to have in the dual automaton all Schreier graphs in the boundary which are either finite, or acyclic multigraphs (just considering the edges without their inverse).

The study of commuting pairs also leads to a connection with periodic tessellations of the discrete plane using Wang tilings, and it is also related to the so-called Gromov's problem (the reader is referred to the paper [21] for more details). This connection has been pointed out to us by I. Bondarenko [5]. Using the helix graph one can easily show that any automaton group has a commuting pair, whence the associated tileset has always a periodic tiling. However this commuting pair may involve a trivial word. For instance in the class \mathcal{S}_a there is always a trivial commuting pair involving the sink-state. This fact leads to the notions of non-elementary commuting pair and reduced tileset of an automaton group. Using a result by [23] we first show that the problem of finding non-elementary commuting pairs is undecidable. Further, the notion of non-elementary commuting pair is strictly related to the existence of periodic singular points in the boundary. From this connection, we start a study of the relationship between non-periodic tessellations of the discrete plane, and algebraic and dynamical properties of the associated automaton group. Indeed, we first provide conditions for the associated reduced tileset to tile the discrete plane. Then, we pinpoint the algebraic and dynamical properties that an automaton group from \mathcal{S}_a has to possess so that the associated reduced tileset generates just aperiodic tilings. Finally, we characterize the existence of aperiodic tilings *à la* Kari-Papasoglu with some properties of the group generated by an automaton and its set of singular points.

2. PRELIMINARIES

2.1. Mealy automata. We first start with some vocabulary on words, then introduce our main tool — Mealy automata.

Let Q be a finite set, as usual, Q^n , $Q^{\leq n}$, $Q^{< n}$, $Q^{\geq n}$, Q^* , and Q^ω denote respectively the set of words of length n , of length less than or equal to n , of length less than n , of length greater than or equal to n , of finite length, and the set of right-infinite words on Q .

For two words $u, v \in Q^*$ with $u = vv'$ ($u = v'v$) for some $v' \in Q^*$, we say that v is a *prefix* (*suffix*), denoted by $v \leq_p u$ (respectively, $v \leq_s u$).

If $\xi = x_1x_2\cdots \in Q^\omega$, then $\xi[n] = x_n$ is the n -th letter of ξ , and $\xi[:n] = x_1\cdots x_n$ its initial prefix of length n . Similarly, for $m \leq n$ we denote by $\xi[m:n] = x_m\cdots x_n$ the factor of ξ of length $n - m + 1$ between the m -th and the n -th letter of ξ , and by $\xi[m:] = x_mx_{m+1}\cdots$ its *tail*. Two infinite sequences $\xi, \eta \in Q^\omega$ are said to be *cofinal* (written $\xi \propto \eta$) if there exists an integer k such that $\xi[k:] = \eta[k:]$.

By $\tilde{Q} = Q \cup Q^{-1}$ we denote the *involutive set* where Q^{-1} is the set of *formal inverses* of Q . The operator $^{-1}: Q \rightarrow Q^{-1}$ sending $q \mapsto q^{-1}$ is extended to an involution on the free monoid \tilde{Q}^* through

$$1^{-1} = 1, \quad (q^{-1})^{-1} = q, \quad (uv)^{-1} = v^{-1}u^{-1} \quad (q \in Q; u, v \in \tilde{Q}^*).$$

Let \sim be the congruence on \tilde{Q}^* generated by the relation set $\{(qq^{-1}, 1) \mid q \in \tilde{Q}\}$. The quotient $F_Q = \tilde{Q}^* / \sim$ is the *free group* on Q , and let $\sigma: \tilde{Q}^* \rightarrow F_Q$ be the canonical homomorphism. The set of all *reduced words* on \tilde{Q}^* may be compactly

written as

$$R_Q = \tilde{Q}^* \setminus \bigcup_{q \in \tilde{Q}} \tilde{Q}^* q q^{-1} \tilde{Q}^*.$$

For each $u \in \tilde{Q}^*$, we denote by $\bar{u} \in R_Q$ the (unique) reduced word \sim -equivalent to u . With a slight abuse in the notation we often identify the elements of F_Q with their reduced representatives, *i.e.* $\sigma(u) = \bar{u}$; this clearly extends to subsets $\sigma(L) = \bar{L}$ for $L \subseteq \tilde{Q}^*$.

A *Mealy automaton* is a tuple $\mathcal{M} = (Q, \Sigma, \cdot, \circ)$ where Q and Σ are finite set respectively called the stateset and the alphabet, and \cdot, \circ are functions from $Q \times \Sigma$ to, respectively, Q and Σ called the transition and the production function. This automaton can be seen as a complete, deterministic, letter-to-letter transducer with same input and output alphabet or, following [10], as a labelled digraph. The graphical representation is standard (see Fig. 2 for instance) and one displays transitions as follows:

$$q \xrightarrow{a|b} p \in \mathcal{M} \iff q \cdot a = p, \quad q \circ a = b.$$

It can be seen that the stateset Q and the alphabet Σ play a symmetric role, hence we can define a new Mealy automaton: the *dual* of the automaton $\mathcal{M} = (Q, \Sigma, \cdot, \circ)$ is the automaton $\mathfrak{d}\mathcal{M} = (\Sigma, Q, \circ, \cdot)$ where we have the transition $a \xrightarrow{q|p} b$ whenever $q \xrightarrow{a|b} p$ is a transition in \mathcal{M} (see Fig. 2).

For each automaton transition $q \xrightarrow{a|q \circ a} q \cdot a$, we associate the *cross-transition* depicted in the following way:

$$\begin{array}{ccc} & a & \\ & \downarrow & \\ q & \xrightarrow{\quad} & q \cdot a, \\ & \uparrow & \\ & q \circ a & \end{array}$$

see also Fig. 1.

If the functions $(\Sigma \rightarrow \Sigma : a \mapsto q \circ a)_{q \in Q}$ are permutations the automaton is said to be *invertible*. On the other hand, when the functions $(Q \rightarrow Q : q \mapsto q \cdot a)_{a \in \Sigma}$ are permutations the automaton is called *reversible*. Note that when an automaton is reversible its dual is reversible and the other way around. Mealy automata that are both invertible and reversible are called reversible invertible automata, or RI-automata for short. Other classes of automata will be described in Section 3.

A Mealy automaton $\mathcal{M} = (Q, \Sigma, \cdot, \circ)$ defines inductively an action $Q \curvearrowright \Sigma^*$ of Q on Σ^* by

$$q \circ (a_1 \cdots a_n) = (q \circ a_1) \circ ((q \cdot a_1) \circ (a_2 \cdots a_n)),$$

that can also be depicted by a cross-diagram by gluing cross-transitions (see Glasner and Mozes [13], or [1]) representing the action of a word of states on a word of letters (or vice-versa):

$$\begin{array}{ccccc} & a_1 & & a_2 \cdots a_n & \\ & \downarrow & & \downarrow & \\ q & \xrightarrow{\quad} & q \cdot a_1 & \xrightarrow{\quad} & (q \cdot a_1) \cdot (a_2 \cdots a_n) \\ & \uparrow & & \uparrow & \\ & q \circ a_1 & & (q \circ a_1) \circ (a_2 \cdots a_n) & \end{array}.$$

In a dual way, this Mealy automaton defines also an action $Q^* \curvearrowright \Sigma$. Both actions naturally extend to words, respectively in Q^* and Σ^* with the convention

$$hg \circ a = h \circ (g \circ a) \text{ and } g \cdot ab = (g \cdot a) \cdot b.$$

In addition to these descriptions of a Mealy automaton, we are going to use another visualization, the *helix graphs* (introduced in [1]). The *helix graph* $\mathcal{H}_{n,k}$ of a Mealy automaton $\mathcal{M} = (Q, \Sigma, \cdot, \circ)$ is the directed graph with nodes $Q^n \times \Sigma^k$ and arcs $(u, v) \rightarrow (u \cdot v, u \circ v)$ for all $(u, v) \in Q^n \times \Sigma^k$ (see Fig. 1).

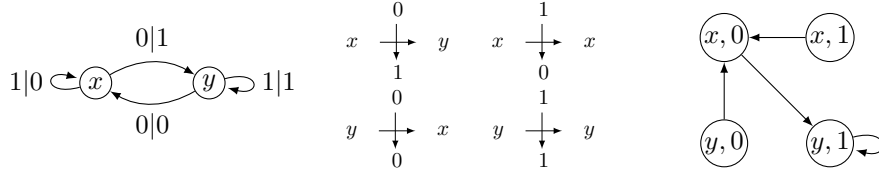


FIGURE 1. The Mealy automaton \mathcal{L} generating the lamplighter group, the set of its cross-transitions, and its helix graph $\mathcal{H}_{1,1}(\mathcal{L})$.

2.2. Automaton groups. From the algebraic point of view, the action $Q^* \curvearrowright \Sigma^*$ gives rise to a semigroup $\langle \mathcal{M} \rangle^+$ generated by the endomorphisms $q \in Q$ of the regular rooted tree identified with Σ^* defined by $q : u \mapsto q \circ u$ for $u \in \Sigma^*$. Groups generated by invertible automata play an important role in group theory (for more details we refer the reader to [25]). In this framework all the maps $q : u \mapsto q \circ u$, $q \in Q$, are automorphisms of the regular rooted tree Σ^* , and the group generated by these automorphisms is denoted by $\langle \mathcal{M} \rangle$ (with identity $\mathbb{1}$). Note that the actions $Q^* \curvearrowright \Sigma^*$ and $Q^* \curvearrowright \Sigma^*$ extend naturally to the actions $\langle \mathcal{M} \rangle \curvearrowright \Sigma^*$ and $\langle \mathcal{M} \rangle \curvearrowright \Sigma^*$, respectively.

There is a natural way to factorize these actions using the wreath product [25, 3].

Let $\mathcal{M} = (Q, \Sigma, \cdot, \circ)$ be an invertible Mealy automaton. The inverse of the automorphism q is denoted by $q^{-1} \in Q^{-1} = \{q^{-1} : q \in Q\}$. There is an explicit way to express the actions of the inverses by considering the inverse automaton \mathcal{M}^{-1} having Q^{-1} as stateset, and a transition $q^{-1} \xrightarrow{b|a} p^{-1}$ whenever $q \xrightarrow{a|b} p$ is a transition in \mathcal{M} (see Fig. 2).

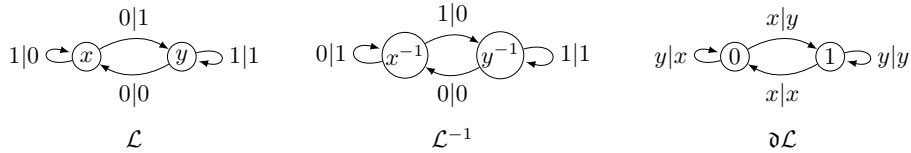


FIGURE 2. The lamplighter automaton \mathcal{L} , its inverse automaton \mathcal{L}^{-1} , and its dual automaton $\mathfrak{d}\mathcal{L}$.

The action of the group $\langle \mathcal{M} \rangle$ on Σ^* , in case \mathcal{M} is invertible (or of the semigroup $\langle \mathcal{M} \rangle^+$ in a more general case), may be naturally extended on the *boundary* Σ^ω of the tree.

This action gives rise to the so-called orbital graph. In general, given a finitely generated semigroup S , with set of generators Q , that acts on the left of a set X according to $S \curvearrowright X$, if $\pi : Q^* \rightarrow S$ denotes the canonical map, then the *orbital graph* $\Gamma(S, Q, X)$ is defined as the Q -digraph with set of vertices X , and there is an edge $x \xrightarrow{a} y$ whenever $\pi(a) \circ x = y$. When we want to pinpoint the connected component containing the element $y \in X$ we use the shorter notation $\Gamma(S, Q, X, y)$ instead of $(\Gamma(S, Q, X), y)$. Note that in the realm of groups, this notion corresponds to the notion of Schreier graph. In particular for an invertible Mealy automaton $\mathcal{M} = (Q, \Sigma, \cdot, \circ)$ and a word $v \in \Sigma^* \sqcup \Sigma^\omega$, if

$$\text{Stab}_{\langle \mathcal{M} \rangle}(v) = \{g \in \langle \mathcal{M} \rangle : g \circ v = v\}$$

is the stabilizer of v , the Schreier graph $\text{Sch}(\text{Stab}_{\langle \mathcal{M} \rangle}(v), \tilde{Q})$ corresponds to the connected component pinpointed by v of the orbital graph:

$$\text{Sch}(\text{Stab}_{\langle \mathcal{M} \rangle}(v), \tilde{Q}) \simeq \Gamma(\langle \mathcal{M} \rangle, \tilde{Q}, \Sigma^* \sqcup \Sigma^\omega, v).$$

This simply corresponds to consider the orbit of v as the vertex set and the edges given by the action of the generators of the group (in our context the state of the generating automaton).

Henceforth, when the automaton group is clear from the context we will use the more compact notation $\text{Sch}(v)$ when we deal with $\text{Sch}(\text{Stab}_{\langle \mathcal{M} \rangle}(v), \tilde{Q})$.

3. THE CONSIDERED CLASSES

Throughout the paper we focus mainly on four classes of automata: *contracting* automata, *reversible* automata, *bireversible* automata, and automata *with a sink*.

3.1. Contracting automata. The notion of contracting automata has been introduced by V. Nekrashevych in [25]. For a Mealy automaton $\mathcal{M} = (Q, \Sigma, \cdot, \circ)$, the group $\langle \mathcal{M} \rangle$ is said to be *contracting* if there exists a finite set $\aleph \subset \langle \mathcal{M} \rangle$ such that, for any $g \in \langle \mathcal{M} \rangle$ there exists an integer $n = n(g)$ such that $g \cdot v \in \aleph$, for any $v \in \Sigma^{\geq n}$. The set \aleph is called a *nucleus* of $\langle \mathcal{M} \rangle$. Graphically it means that, from any element of the group, a long enough path leads to the nucleus. This enables the construction of a finite automaton \mathcal{M}_\aleph with stateset \aleph , alphabet Σ , and transitions $g \xrightarrow{a|g \circ a} g \cdot a$. By extension, an automaton generating a contracting group is said to be *contracting* itself. Examples of such automata are depicted on Fig. 3 and 4.

Note that if the contracting automaton \mathcal{M} has a sink-state, this state necessarily belongs to \aleph .

The importance of the notion of contracting automata refers to the beautiful and surprising connection with complex dynamics established by V. Nekrashevych [25]. With every contracting group one may associate a topological space called *limit space*, that is encoded by the set of left-infinite words on Σ modulo the equivalence relation given by the action of the nucleus, *i.e.*, two left infinite sequences $\xi = \cdots \xi_n \xi_{n-1} \cdots \xi_1$ and $\eta = \cdots \eta_n \eta_{n-1} \cdots \eta_1$ are equivalent if for any $n \geq 1$ there exists $g_n \in \aleph$ satisfying $g_n \circ \xi_n \xi_{n-1} \cdots \xi_1 = \eta_n \eta_{n-1} \cdots \eta_1$. It turns out that the iterated monodromy group $\text{IMG}(f)$ of a post-singularly finite rational function is contracting and its limit space is homeomorphic to the Julia set of f . This discovery puts in strict relation the dynamics of the map f and the algebraic properties of $\text{IMG}(f)$. As an example, this powerful correspondence has allowed L. Bartholdi

and V. Nekrashevych to solve a classical problem in complex dynamics, the so-called Hubbard Twisted Rabbit Problem by using algebraic methods [2].

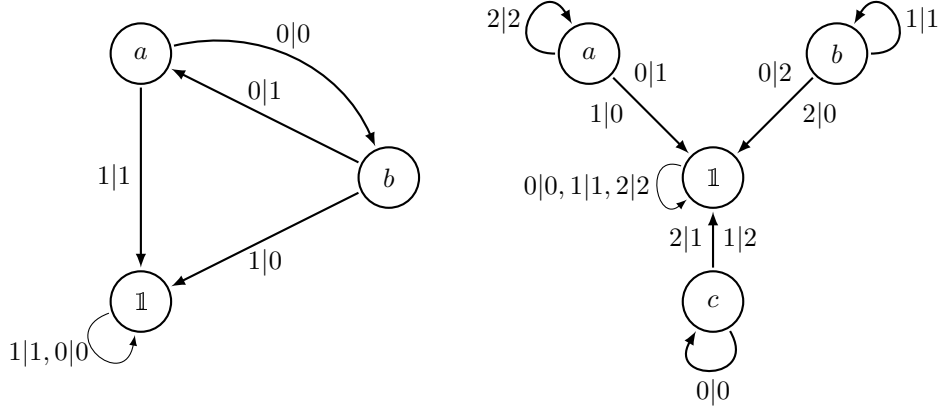


FIGURE 3. The contracting automata generating the Basilica group (on the left) and the Hanoi Towers group $H^{(3)}$ (on the right).

3.2. (Bi)reversible automata. The classes of reversible and bireversible Mealy automata are also interesting. We recall that a Mealy automaton is *reversible* whenever each input letter induces a permutation of the stateset, *i.e.* simultaneous transitions $q \xrightarrow{a|b} p$ and $q' \xrightarrow{a|c} p$ are forbidden. In the context of groups, we are especially interested in *reversible invertible* automata (called henceforward *RI*-automata).

Moreover such a reversible automaton is *bireversible* if in addition each output letter induces a permutation of the stateset, *i.e.* simultaneous transitions $q \xrightarrow{a|b} p$ and $q' \xrightarrow{c|b} p$ are also forbidden. In this case it is necessarily invertible.

An interesting feature of an *RI*-automaton is that the dual of such an automaton is still reversible and invertible.

The following lemma, which will be useful later, may be easily deduced from [32] or [10, Theorem 2].

Lemma 3.1. *Let \mathcal{M} be an RI-automaton with $\langle \mathcal{M} \rangle \simeq F_Q/N$. Then, the following facts hold:*

- (i) *if $g \in Q^*$ is such that $g \cdot a \in N$ for some $a \in \Sigma^*$, then $g \in N$;*
- (ii) *let $g, g', g'' \in Q^*$ with $g \cdot a = g'$ for some $a \in \Sigma^*$, then there is a $h \in Q^*$ satisfying $(hg) \cdot a = g''g'$.*

Furthermore, if \mathcal{M} is bireversible then Q can be replaced by \tilde{Q} in (i).

Proof. Let us prove point (i). First note that any state reachable from a state in N also belongs to N . Indeed let $g' \in N$ and $a', b' \in \Sigma^*$, we have:

$$\begin{array}{ccc} a' & & b' \\ g' \downarrow & \rightarrow & g' \cdot a' \downarrow \\ a' & & b' \end{array}$$

So for any word $b' \in \Sigma^*$, $g' \circ a' b' = a' b'$, hence $(g' \cdot a') \circ b' = b'$, i.e., $g' \cdot a' \in N$.

Let $g' = g \cdot a \in N$. By the reversibility of the automaton there is an $a' \in \Sigma^*$ such that $g' \cdot a' = g$ and we can conclude that g belongs to N .

Property (ii) follows by observing that by reversibility there is an $h \in Q^*$ such that $h \cdot (g \circ a) = g''$ holds (remember that the words in Q^* are written from right to left). In terms of cross-diagrams we obtain:

$$\begin{array}{ccc} & a & \\ & \downarrow & \\ g & \xrightarrow{\quad} & g' = g \cdot a \\ & \downarrow & \\ & g \circ a & \\ h & \xrightarrow{\quad} & g'' = h \cdot (g \circ a) \end{array} \quad .$$

The last statement follows by applying (i) to $\mathcal{M} \sqcup \mathcal{M}^{-1}$ that is reversible by the bireversibility of \mathcal{M} . \square

The structure of the groups generated by reversible or bireversible automata is far from being understood. For instance, for a long time the only known examples of groups generated by bireversible automata were finite, free, or free products of finite groups [34, 35, 25]. Recently examples of bireversible automata generating non-finitely presented groups have been exhibited in [7] and in [22, 29]. In this regard, we now provide an embedding result of any group generated by a bireversible automaton whose dual does not generate a free group, into the outer automorphism group of a free group of infinite rank. This fact may give some extra insight on the kind of groups that are generated considering bireversibility.

Proposition 3.2. *Let \mathcal{M} be a bireversible automaton such that $\langle \mathcal{M} \rangle$ is infinite and $\langle \mathfrak{d}\mathcal{M} \rangle$ is not free. Then, there is a monomorphism*

$$\phi : \langle \mathcal{M} \rangle \hookrightarrow \text{Out}(F_\infty).$$

Proof. Let $\mathcal{M} = (Q, \Sigma, \cdot, \circ)$. By [8, Theorem 4], if we consider the enriched automaton \mathcal{M}^- obtained from \mathcal{M} by adding the edge $p \xrightarrow{a|b} q$ for any edge $q \xrightarrow{a|b} p$ of \mathcal{M} , then $\langle \mathcal{M}^- \rangle \simeq \langle \mathcal{M} \rangle$. By [10, Theorem 2] we may express the group $\langle \mathfrak{d}\mathcal{M} \rangle$ as the quotient F_Σ / \overline{N} where N is the maximal subset invariant under the action $Q \curvearrowright \tilde{\Sigma}^*$. We may regard N as a normal subgroup of the free group F_Σ , in particular note that $[F_\Sigma : N] < \infty$ if and only if $\langle \mathfrak{d}\mathcal{M} \rangle$ is finite, and so $\langle \mathcal{M} \rangle$ is also finite [25, 30]. Therefore, $[F_\Sigma : N] = \infty$, whence $N \simeq F_\infty$, since N is free by Nielsen's theorem. Let us first prove that there is an embedding $\langle \mathcal{M} \rangle \hookrightarrow \text{Aut}(N)$. By the stability of N under the action $Q \curvearrowright \tilde{\Sigma}^*$ and the invertibility of \mathcal{M}^- , we have that for any $g \in \langle \mathcal{M} \rangle$ the map $\psi_g : n \mapsto g \circ n$ for $n \in N$ is a bijection of N that is also a homomorphism since $w \cdot n = w$ holds for any $w \in \tilde{Q}^*$ and $n \in N$. Hence, $\psi_g \in \text{Aut}(N)$. Furthermore, the map $\phi : \langle \mathcal{M} \rangle \rightarrow \text{Aut}(N)$ that sends g to ψ_g is a homomorphism since equality $\psi_{g'g} = \psi_{g'} \circ \psi_g$ holds. This map is also injective. Indeed, assume that $\psi_{g_1} = \psi_{g_2}$ for some $g_1, g_2 \in \langle \mathcal{M} \rangle$. Since N is normal, then for any $u \in \Sigma^*$ there is a reduced element $n \in N$ such that u is a prefix of n (take a suitable conjugate of a reduced non-trivial element of N). Hence, $\psi_{g_1} = \psi_{g_2}$ implies that $g_1 \circ u = g_2 \circ u$ for any $u \in \Sigma^*$, hence $g_1 = g_2$. Hence we get the claim $\phi : \langle \mathcal{M} \rangle \hookrightarrow \text{Aut}(N)$. We now show that each automorphism ψ_g is not inner. Indeed, assume contrary to our claim, that $\psi_g(n) = unu^{-1}$, $n \in N$, for some reduced non-empty element $u \in N$.

Then, for any $n \in N$ there is an integer $\ell(n)$ such that $g^{\ell(n)} \circ n = n$, whence we have

$$n = \psi_{g^{\ell(n)}}(n) = \psi_g^{\ell(n)}(n) = u^{\ell(n)} n u^{-\ell(n)}$$

from which we get $nu^{\ell(n)} = u^{\ell(n)}n$. We consider the subgroup generated by n and $u^{\ell(n)}$. By Nielsen's theorem this subgroup is free, and so both n and $u^{\ell(n)}$ belongs to the same cyclic subgroup $\langle h \rangle$ for some $h \in N$. In particular, there is a non-empty prefix $h' \in \tilde{\Sigma}^*$ common to both u and n . However, since $n \in N$ is arbitrary and N is normal by the same argument above each $u \in \tilde{\Sigma}^*$ appears as a prefix of some non-trivial reduced element $n' \in N$, a contradiction. Hence, $\phi : \langle \mathcal{M} \rangle \hookrightarrow \text{Out}(N) \simeq \text{Out}(F_\infty)$. \square

Unfortunately, the condition of having a free group of infinite rank appears to be mandatory in Proposition 3.2. Indeed, the next proposition shows that the embedding of an automaton group generated by a bireversible automaton into the group of length preserving automorphisms of the free group F_m for some $1 < m < \infty$, characterizes finite groups. In what follows we call an element $\psi \in \text{Aut}(F_m)$ *length preserving* if given any $w \in F_m$ one has $|w| = |\psi(w)|$. We denote by $\text{Aut}_\ell(F_m)$ the subgroup of $\text{Aut}(F_m)$ formed by the length preserving automorphisms.

Proposition 3.3. *Let \mathcal{M} be a bireversible automaton. There is a monomorphism*

$$\phi : \langle \mathcal{M} \rangle \hookrightarrow \text{Aut}_\ell(F_m)$$

if and only if $\langle \mathcal{M} \rangle$ is finite.

Proof. Let $\mathcal{M} = (Q, \Sigma, \cdot, \circ)$. Suppose $\langle \mathcal{M} \rangle$ finite and let N be the maximal invariant subset for the action $Q \curvearrowright \tilde{\Sigma}^*$ (as in Proposition 3.2), then $[F_\Sigma : N] < \infty$ and so $N \simeq F_m$ for some m . We proceed as in the proof of the previous proposition to show that $\phi : \langle \mathcal{M} \rangle \rightarrow \text{Aut}_\ell(F_m)$ is a monomorphism. Conversely, let consider an embedding $\phi : \langle \mathcal{M} \rangle \hookrightarrow \text{Aut}_\ell(F_m)$ for some m , and let $R = \{x_1, \dots, x_m\}$ be minimal set of generators of F_m . For any $g \in \langle \mathcal{M} \rangle$, let ψ_g be the corresponding automorphism in $\text{Aut}_\ell(F_m)$. Since R is finite and the automorphisms preserve the length, the set

$$\Omega = \bigcup_{g \in \langle \mathcal{M} \rangle, i=1, \dots, m} \psi_g(x_i)$$

is clearly finite. Further, there is a natural homomorphism of $\langle \mathcal{M} \rangle$ into $\text{Sym}(\Omega)$. Let us prove that it is actually a monomorphism. Indeed, let $g \neq g'$ in $\langle \mathcal{M} \rangle$. Then $\psi_g \neq \psi_{g'}$ holds in $\text{Aut}_\ell(F_m)$. Since R is a generating set, then $\psi_g(x_{i_k}) \neq \psi_{g'}(x_{i_k})$ for some $x_{i_k} \in R$. We deduce $\langle \mathcal{M} \rangle \hookrightarrow \text{Sym}(\Omega)$, and so $\langle \mathcal{M} \rangle$ is finite. \square

For similar results that link automaton groups defined by bireversible Mealy automata and the group of automorphisms of a free group, the reader is referred to [24].

3.3. Automata with sink. In the complement of the class of the *RI*-automata there is another interesting class that, in some sense, represents the opposite case: the class \mathcal{S}_a of all the invertible Mealy automata with a sink-state e which is accessible from every state (the index “a” standing for accessible). We recall that a *sink-state* of a Mealy automaton is a special state e such that $e \cdot a = e$ and $e \circ a = a$ for any $a \in A$. Note that in this setting the sink-state is unique. The reason we require that the sink-state is accessible from every state will be clear

in Section 6.

This class is rather broad and it contains many known classes of Mealy automata like automata with polynomial state activity [31]. Furthermore, in [10, Proposition 6] it is shown that this class is essentially formed by those automata for which every element g in the generated group has a g -regular element in the boundary (for the notion of g -regular element see for instance [26]).

Moreover, this class is also included into the broader class of *synchronizing* automata for which some results on automaton groups can be found in [9]. The connection with synchronizing automata will also be crucial in Section 6 in characterizing automata whose associated set of reduced tiles do not tile the plane. In [8] the problem of finding free groups generated by automata in \mathcal{S}_a is tackled. Indeed, until recently, all known free automaton groups were generated by bireversible automata. This led to the question whether or not it is possible to generate a free group by means of automata with a sink-state. In [8] a series of examples of automata from \mathcal{S}_a generating free groups is exhibited. However, in this case the resulting free groups do not act transitively on the corresponding tree, so this leaves open the question of finding a free group generated by an automaton from \mathcal{S}_a acting transitively on the rooted tree. This problem is also connected with the interesting combinatorial notion of *fragile word* introduced in [8].

4. TOPOLOGICAL PROPERTIES OF THE ACTION ON THE BOUNDARY

In this section we describe some topological properties of the action of an automaton group on the boundary of a rooted tree. In particular, we consider the problem of continuity of the map that associates with any point in the boundary the corresponding Schreier graph. We prove that the set of those points where this function is not continuous has zero measure. Moreover, we provide a characterization of contracting automata whose action on the boundary is continuous everywhere. In the reversible case, we prove that examples of automata generating groups with all continuous points in the boundary are necessarily bireversible, and in this case, this condition may be rephrased in terms of triviality of the stabilizers in the boundary.

4.1. Action on the boundary. Let $\mathcal{M} = (Q, \Sigma, \cdot, \circ)$ be an invertible automaton and $Sub(\langle \mathcal{M} \rangle)$ denote the space of all subgroups of $\langle \mathcal{M} \rangle$ and let $Sch(\langle \mathcal{M} \rangle, \tilde{Q})$ denote the space of marked Schreier graphs of $\langle \mathcal{M} \rangle$ (i.e. Schreier graphs in which we have chosen a special vertex, the marked vertex) contained in the space of all marked labeled graphs and put $\partial T = \Sigma^\omega$. Both spaces may be endowed with a natural topology (also induced by an opportune metric). We endow the space $Sub(\langle \mathcal{M} \rangle)$ with the Tikhonov topology of the space $\{0, 1\}^{\langle \mathcal{M} \rangle}$ in such a way that any subgroup H may be identified with its characteristic function. Given a finite subset F of $\langle \mathcal{M} \rangle$ the F -neighborhood of a subgroup H contains all subgroups K such that $H \cap F = K \cap F$. Roughly speaking we say that two subgroups H and K of $\langle \mathcal{M} \rangle$ are close if they share many elements. On the other hand, two marked Schreier graphs $Sch(\xi)$ and $Sch(\eta)$ are close when the subgraphs given by the balls of large radius around ξ and η are isomorphic, and two points ξ and η in ∂T are close if they share a long common prefix. Notice that in our notation, $Sch(\xi)$ corresponds

to the graph $\Gamma(\langle \mathcal{M} \rangle, \tilde{Q}, \Sigma^* \sqcup \Sigma^\omega, \xi)$.

Vorobets studied the map

$$\begin{aligned} F: \partial T &\longrightarrow \text{Sch}(\langle \mathcal{M} \rangle, S) \\ \xi &\longmapsto \text{Sch}(\text{Stab}_{\langle \mathcal{M} \rangle}(\xi), \tilde{Q}) \end{aligned}$$

in the case where \mathcal{M} is the Grigorchuk automaton [36]. His results may be summarized as follows: the closure $\overline{F(\partial T)}$ of the image of the boundary of the binary tree into the space of marked labeled Schreier graphs consists of a countable set of points (the one-ended boundary graphs) and another component containing all two-ended Schreier graphs. The Grigorchuk group acts on the compact component given by $\overline{F(\partial T)}$ without these isolated points by shifting the marked vertex of the graph and such action is minimal (every orbit is dense) and uniquely ergodic (there is a unique Borel probability measure on this set that is invariant under the action of the group).

4.2. Singular points. Let $\mathcal{M} = (Q, \Sigma, \cdot, \circ)$ invertible. We define the map

$$\begin{aligned} \text{St}: \Sigma^\omega &\longrightarrow \text{Sub}(\langle \mathcal{M} \rangle) \\ \xi &\longmapsto \text{Stab}_{\langle \mathcal{M} \rangle}(\xi). \end{aligned}$$

The *neighborhood stabilizer* $\text{Stab}_{\langle \mathcal{M} \rangle}^0(\xi)$ of ξ is the set of all $g \in \langle \mathcal{M} \rangle$ that fix the point ξ together with its neighborhood (that may depend on g). One may check that $\text{Stab}_{\langle \mathcal{M} \rangle}^0(\xi)$ is a normal subgroup of $\text{Stab}_{\langle \mathcal{M} \rangle}(\xi)$.

A point $\xi \in \Sigma^\omega$ is called *singular* if the map St is not continuous at ξ . The set of singular points is denoted by κ .

The following lemma clarifies the connection between the continuity of the map St and the dynamics in the boundary.

Lemma 4.1. [36, Lemma 5.4] *St is continuous at the point ξ if and only if the stabilizer of ξ under the action coincides with its neighborhood stabilizer, i.e.:*

$$\xi \in \kappa \iff \text{Stab}_{\langle \mathcal{M} \rangle}^0(\xi) \neq \text{Stab}_{\langle \mathcal{M} \rangle}(\xi).$$

The following lemma characterizes continuous points in terms of restrictions.

Lemma 4.2. *Let \mathcal{M} be an invertible automaton and let ξ be an element in Σ^ω . The following are equivalent.*

- (i) ξ is not singular;
- (ii) St is continuous at ξ ;
- (iii) For any $g \in \text{Stab}_{\langle \mathcal{M} \rangle}(\xi)$ there exists n such that $g \cdot \xi[:n] = 1$;

Proof. (i) \Leftrightarrow (ii), (iii) \Rightarrow (ii), follow from definition.

(ii) \Rightarrow (iii). Let us prove $\neg(\text{iii}) \Rightarrow \neg(\text{ii})$. Suppose that there exists $g \in \text{Stab}_{\langle \mathcal{M} \rangle}(\xi)$ such that $g \cdot \xi[:n] \neq 1$ for all $n \geq 0$. Then we can find, for any n , a letter $x_n \in \Sigma$ such that $(g \cdot \xi[:n]) \circ x_n = x'_n \neq x_n$. Hence if we put $\zeta_n = \xi[:n]x_n\xi[n+2:] \in \Sigma^\omega$ we get $g \circ \zeta_n = g \circ \xi[:n]x_n\xi[n+2:] = \xi[:n]x'_n\xi[n+2:] \neq \xi$. Hence, since we can construct a ζ_n in any neighbourhood of ξ , $\text{Stab}_{\langle \mathcal{M} \rangle}^0(v^\omega) \neq \text{Stab}_{\langle \mathcal{M} \rangle}(v^\omega)$, and St is not continuous. \square

Moreover we can characterize continuous points by looking only to periodic points

Lemma 4.3. *Let \mathcal{M} be an invertible automaton. The following are equivalent.*

- (i) *There is no singular point in Σ^ω ;*
- (ii) *There is no singular periodic point in Σ^ω .*

Proof. (i) \Rightarrow (ii) is obvious, let us prove the converse, by contraposition. Assume that St is not continuous at some $\xi \in \Sigma^\omega$ and let $g \in \text{Stab}_{\langle \mathcal{M} \rangle}(\xi) \setminus \text{Stab}_{\langle \mathcal{M} \rangle}^0(\xi)$. If there exists k such that $g \cdot \xi[:k] = \mathbb{1}$ then g stabilizes some neighborhood U of ξ and is contained in $\text{Stab}_{\langle \mathcal{M} \rangle}^0(\xi)$. Hence, for any $n \geq 1$, $g \cdot \xi[:n]$ is a non-trivial element in $\langle \mathcal{M} \rangle$. The set $\{g \cdot \xi[:k], k \in \mathbb{N}\}$ is finite, this implies that there exist m and n such that $n > m > 0$ and $g' := g \cdot \xi[:m] = g \cdot \xi[:n] \neq \mathbb{1}$. Therefore $g' \cdot \xi[m+1:n] = g'$ and $g' \circ \xi[m+1:n] = \xi[m+1:n]$. Put $v = \xi[m+1:n]$: $g' \in \text{Stab}_{\langle \mathcal{M} \rangle}(v^\omega)$. In order to prove that $g' \notin \text{Stab}_{\langle \mathcal{M} \rangle}^0(\xi)$ we notice that, since g' is not trivial, there exists $w \in \Sigma^*$ such that $g' \circ w = w' \neq w$. Consider the sequence $w_k := v^k w v^\omega$ for $k \geq 0$. Clearly, for any neighborhood U of v^ω there exists n such that $w_n \in U$. But

$$g' \circ w_n = g' \circ v^n (g' \cdot v^n) \circ (w v^\omega) = v^n w' v' \neq w_n$$

for some $v' \in \Sigma^\omega$. Therefore $g' \in \text{Stab}_{\langle \mathcal{M} \rangle}(v^\omega) \neq \text{Stab}_{\langle \mathcal{M} \rangle}^0(v^\omega)$. \square

In the following theorem we prove that the measure of the set κ of singular points is zero. For the sake of completeness we recall that a subset of a topological space X is nowhere dense if its closure has an empty interior. A subset is *meager* in X if it is a union of countably many nowhere dense subsets. A Baire space, as Σ^ω with the usual topology, cannot be given by the countable union of disjoint nowhere dense sets. In general the notion of nowhere dense and meager set do not coincide with the notion of zero-measure. In [36] it is proven that κ is meager.

Given $u \in \tilde{Q}^*$, denote by $\text{Fix}(u)$ the set consisting in the vertices $\xi \in \Sigma^\omega$ fixed by the action of u . If $w \in \Sigma^k$ is an element stabilized by u we write $w \in \text{Fix}_k(u)$. By using the ideas developed in [20] we are able to give the following characterization.

Theorem 4.4. *For any invertible automaton, the set κ of singular points has measure zero.*

Proof. The proof is heavily based on the ideas contained in Proposition 4.1 and Theorem 4.2 of [20]. For $u \in \tilde{Q}^*$, $k \geq 1$, consider the following sets:

$$\begin{aligned} \chi(u) &= \{\xi \in \Sigma^\omega : \xi \in \text{Fix}(u) \text{ and } \pi(u \cdot \xi[:j]) \neq \mathbb{1} \quad \forall j \geq 0\}, \\ \chi_k(u) &= \{w \in \Sigma^k : w \in \text{Fix}_k(u) \text{ and } \pi(u \cdot \xi[:j]) \neq \mathbb{1} \quad 0 \leq j \leq k\}. \end{aligned}$$

By Lemma 4.2, the set of singular points is

$$\kappa = \bigcup_{u \in \tilde{Q}^*} \chi(u).$$

Let us prove $\mu(\chi(u)) = 0$, for any $u \in \tilde{Q}^*$. Since

$$\chi(u) = \bigcap_{k \geq 1} (\chi_k(u) \Sigma^\omega),$$

we get:

$$\mu(\chi(u)) = \lim_{k \rightarrow \infty} \mu(\chi_k(u) \Sigma^\omega) = \lim_{k \rightarrow \infty} \frac{|\chi_k(u)|}{|\Sigma|^k}.$$

We now show that this limit is 0. Indeed, as proved below, there is an integer p such that

$$(E_k) \quad |\chi_{pk}(u)| \leq (|\Sigma|^p - 1)^k, \text{ for all } k \geq 1.$$

Let $H_i = \{v \in \tilde{Q}^i : \pi(v) \neq 1\}$. Since $H_{|u|}$ is finite, there is an integer p such that no element of $H_{|u|}$ induces the identity on Σ^p : take it for Equation (E_k) .

Use an induction on $k \geq 1$. For $k = 1$: $\chi_p(u) \subseteq \Sigma^p$ and $\text{Fix}_p(u) \neq \Sigma^p$ by the choice of p . Suppose that $|\chi_{p(k-1)}(u)| \leq (|\Sigma|^p - 1)^{k-1}$. Since $u \cdot h \in H_{|u|}$ for any $h \in \chi_{p(k-1)}(u)$, there is a $v \in \Sigma^p$ that is not fixed by $u \cdot h$, whence

$$|\chi_{p \cdot k}(u)| \leq |\chi_{p(k-1)}(u)|(|\Sigma|^p - 1) \leq (|\Sigma|^p - 1)^k.$$

□

4.3. The contracting case. The property of being contracting allows us to characterize the set κ of singular points in terms of a language recognized by an automaton.

Let $\mathcal{M} = (Q, \Sigma, \cdot, \circ)$ be a Mealy automaton. We define its *stable automaton* as the automaton $\mathcal{B}(\mathcal{M})$ on infinite words where for $a \in \Sigma$ we have:

$$q \xrightarrow{a} p \in \mathcal{B}(\mathcal{M}) \iff q \xrightarrow{a|a} p \in \mathcal{M}.$$

Given a Büchi acceptance condition (*i.e.* a set of states that has to be visited infinity often), such an automaton recognizes a language of right-infinite words (see for instance [33]). See Figs. 4 and 5.

Theorem 4.5. *Let \mathcal{M} be an invertible automaton admitting a finite nucleus automaton \mathcal{N} . Then the set κ of singular points is included in the set of word cofinal with a word in the language recognized by the Büchi automaton $\mathcal{B}(\mathcal{N})$ with every state but the sink-state accepting. Conversely any word recognized by $\mathcal{B}(\mathcal{N})$ is singular. In particular if the language $\mathcal{B}(\mathcal{N})$ is empty then $\kappa = \emptyset$.*

Proof. Let e be the sink-state of the nucleus and suppose that there exists $\xi \in \Sigma^\omega$ such that St is not continuous on ξ . From Lemma 4.2 it follows that there exists $g \in \text{Stab}_{\langle \mathcal{M} \rangle}(\xi)$ such that $g \cdot \xi[:k] \neq e$ for every $k > 0$. Since $\langle \mathcal{M} \rangle$ is contracting there exists $g_n \in \mathcal{N}$ such that $g_n = g \cdot \xi[:n] \in \mathcal{N}$. Notice that $g_{n+k} := g \cdot \xi[:n+k] = g_n \cdot \xi[n+1 : n+k] \in \mathcal{N} \setminus \{e\}$ and $g_{n+k} \circ \xi[n+k] = \xi[n+k]$. Since $\mathcal{B}(\mathcal{N})$ has the same set of states as \mathcal{N} and admits a transition exactly when $g \circ i = i$ holds, there is an infinite path

$$g_n \xrightarrow{\xi[n]} g_{n+1} \xrightarrow{\xi[n+1]} \dots g_{n+k} \xrightarrow{\xi[n+k]} \dots$$

that avoids e . Furthermore since the stateset is finite there is a state that is infinitely visited, hence the run is accepted by $\mathcal{B}(\mathcal{N})$. To prove that ξ is cofinal to a word recognized by $\mathcal{B}(\mathcal{N})$ notice that since $\xi[n:]$ is recognized and \mathcal{N} is finite, there exists an integer m such that $g \cdot \xi[:m] = g_{m+1}$ belongs to a strongly connected component \mathcal{C} of \mathcal{N} . Hence, by reading transition in \mathcal{C} backward, we can extend $\xi[m:]$ on the left to obtain a word $\xi' = \xi'[1] \dots \xi'[m-1] \xi[m:]$ and a sequence g', g'_1, \dots, g'_m of elements of \mathcal{C} such that there exist $g' \cdot \xi'[:k] = g'_{k+1}$ for $k \leq m$, $g' \cdot \xi'[:m] = g_{m+1}$ and $g' \circ \xi'[:k] = \xi$. Then $\mathcal{B}(\mathcal{N})$ recognizes ξ' , and the points ξ' and ξ are cofinal. On the other hand if a run $g_1 \xrightarrow{i_1} g_2 \dots g_k \xrightarrow{i_k} g_{k+1} \dots$ is accepted by $\mathcal{B}(\mathcal{N})$ then it is infinite and it does not visit e (since it is a sink, *i.e.* an absorbing state). Hence g_1 stabilizes $\xi = i_1 i_2 \dots$ and satisfies $\pi(g_1) \neq 1$. So by Lemma 4.2, St is not continuous. □

Notice that if $\xi \in \Sigma^\omega$ is singular, so is each element of its orbit. Indeed if $g \circ \xi = \xi$ with $g \neq 1$ then we have $hgh^{-1} \circ (h \circ \xi) = h \circ \xi$ with $hgh^{-1} \neq 1$ for all $h \in \langle \mathcal{M} \rangle$. So the previous characterization is exact when the set of words cofinal to a word in the language $\mathcal{B}(\mathcal{N})$ coincide with the orbit of words in the language $\mathcal{B}(\mathcal{N})$.

In particular this occurs for self-replicating automata. A Mealy automaton is called *self-replicating* (also called *fractal*) whenever for any word $u \in \Sigma^*$ and any $g \in \langle \mathcal{M} \rangle$, there exists $h \in \text{Stab}_{\langle \mathcal{M} \rangle}(u)$ satisfying $h \cdot u = g$.

Proposition 4.6. *Let \mathcal{M} be a contracting, self-replicating Mealy automaton. Then κ is the set of words cofinal to a word in the language recognized by $\mathcal{B}(\mathcal{N})$.*

Proof. The first inclusion comes directly from Theorem 4.5. For the other one, let ξ be recognized by $\mathcal{B}(\mathcal{M})$ and $u\xi[n:]$ be cofinal to ξ . Since $\xi[n:]$ is recognized by $\mathcal{B}(\mathcal{N})$ it is singular. So there exists $g \in \langle \mathcal{M} \rangle$ satisfying $g \circ \xi[n:] = \xi[n:]$ with $g \cdot \xi[n:n+k] \neq \mathbb{1}$ for all $k \geq 0$. Now, since \mathcal{M} is self-replicating there exists $h \in \langle \mathcal{M} \rangle$ satisfying the following cross diagram:

$$\begin{array}{ccc} & u & \xi[n:] \\ h \downarrow & \rightarrow & h \cdot u = g \rightarrow \\ & u & \xi[n:] \end{array}$$

Hence $u\xi$ is singular. \square

Note that, despite of these strong requirements, the class of contracting and self-replicating automata is wide. It contains in particular the Grigorchuk automaton, the Hanoi Tower automaton or the basilica automaton.

One can notice that all states in a connected component have same type (accepting or rejecting). In that case the automaton is said to be *weak Büchi* [27]. It follows that the language of $\mathcal{B}(\mathcal{N})$ is closed and regular. It is also (infinite) suffix closed.

Moreover we get informations about the topology of the border $F(\partial T)$.

Proposition 4.7. *Let \mathcal{M} be a Mealy automaton and a be a letter in its alphabet. If there exists a state q such that $q \circ a = a$ and $q \cdot a = q$, and $p \cdot a \neq p$ for $p \neq q \in Q$, then $F(a^\omega) = \text{Sch}(a^\omega)$ is an isolated point in the closure of $F(\partial T)$.*

Proof. Recall that two rooted graphs are close if they have balls of large radius around the roots that are isomorphic. Let us prove $\text{Stab}_{\langle \mathcal{M} \rangle}(a^\omega) \neq \text{Stab}_{\langle \mathcal{M} \rangle}^0(a^\omega)$. The Schreier graph associated with a^ω contains a loop rooted at a^ω and labeled by q . Any other element in ∂T that is not cofinal to a^ω , contains at some position a letter other than a , so that it is not fixed by q . This implies that $F(a^\omega)$ is isolated, since no other graph contains a loop labeled by q .

The same argument works for the vertices of the orbit of a^ω , that are the only ones having at finite distance a vertex with a loop labeled by q . \square

Note that this is still true when one consider words on Σ^k instead of single letters.

We now apply the previous characterizations to two examples of contracting, self-replicating automata, namely the Basilica automaton et the Hanoi Towers automaton.

The Basilica group, introduced in [18], is generated by the automorphisms a and b having the following self-similar form:

$$a = (b, \mathbb{1}), \quad b = (a, \mathbb{1})(01),$$

where (01) denotes the nontrivial permutation of the symmetric group on $\{0, 1, 2\}$ and, with a slightly abuse of notation, $\mathbb{1}$ denotes the sink-state (see Fig. 3). In Fig. 4 the nucleus automaton associated with this group is presented.

Since the stable automaton accepts no infinite word we obtain:

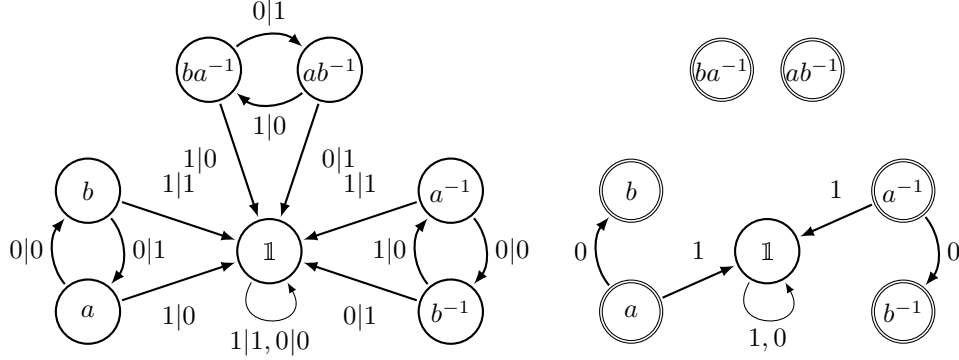


FIGURE 4. The nucleus automaton (on the left) and the stable automaton (on the right), associated with the Basilica automaton.

Corollary 4.8. *For the Basilica group, the set κ of singular points is empty.*

We now consider the case of the Hanoi Towers group $H^{(3)}$ (see Fig. 3) introduced in [17]. This group is generated by the automorphisms of the ternary rooted tree having the following self-similar form:

$$a = (\mathbb{1}, \mathbb{1}, a)(01) \quad b = (\mathbb{1}, b, \mathbb{1})(02), \quad \text{and} \quad c = (c, \mathbb{1}, \mathbb{1})(12),$$

where (01), (02), and (12) are elements of the symmetric group on $\{0, 1, 2\}$. Observe that a, b, c are involutions.

Corollary 4.9. *For the Hanoi Towers group $H^{(3)}$, the set κ of singular points is a countable set consisting of the (disjoint union of the) orbits of the three points 0^ω , 1^ω , and 2^ω . Moreover $F(i^\omega) = \text{Sch}(i^\omega)$ is an isolated point in the closure of $F(\partial T)$ for $i \in \{0, 1, 2\}$.*

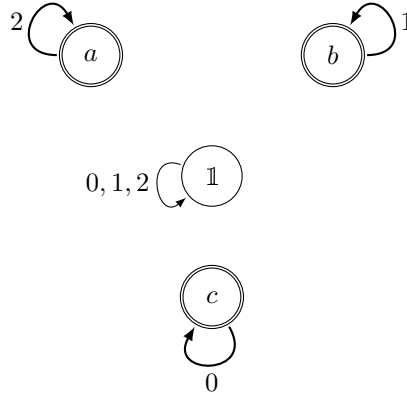


FIGURE 5. The stable automaton $\mathcal{B}(H^{(3)})$ for the Hanoi Towers automaton.

Since $\text{Sch}(i^\omega)$ is isolated in $F(\partial T)$ for $i \in \{0, 1, 2\}$, one can ask about the behaviour of the sequence of Schreier graphs of finite words converging to i^ω . It turns out that this sequence is converging, but the limit is not a Schreier graph for the Hanoi Towers group.

Let $(\eta_n^i)_{n \in \mathbb{N}}$, $i = 0, 1, 2$ be a sequence of elements of Σ^* , such that $|\eta_n^i| = n$, converging to i^ω . Recall that $\text{Sch}(i^\omega)$ contains only one loop rooted at i^ω .

Let Υ_i , $i = 0, 1, 2$ be the graph obtained from $\text{Sch}(i^\omega)$ as follows:

- (1) Take two copies of $\text{Sch}(i^\omega)$ and let $g_i \in \{a, b, c\}$ be the label of the loop at i^ω .
- (2) Erase the loop at i^ω in each copy of $\text{Sch}(i^\omega)$.
- (3) Join the two copies by an edge labeled g_i and connecting the vertices i^ω of each copy and choose one of these i^ω as marked vertex.

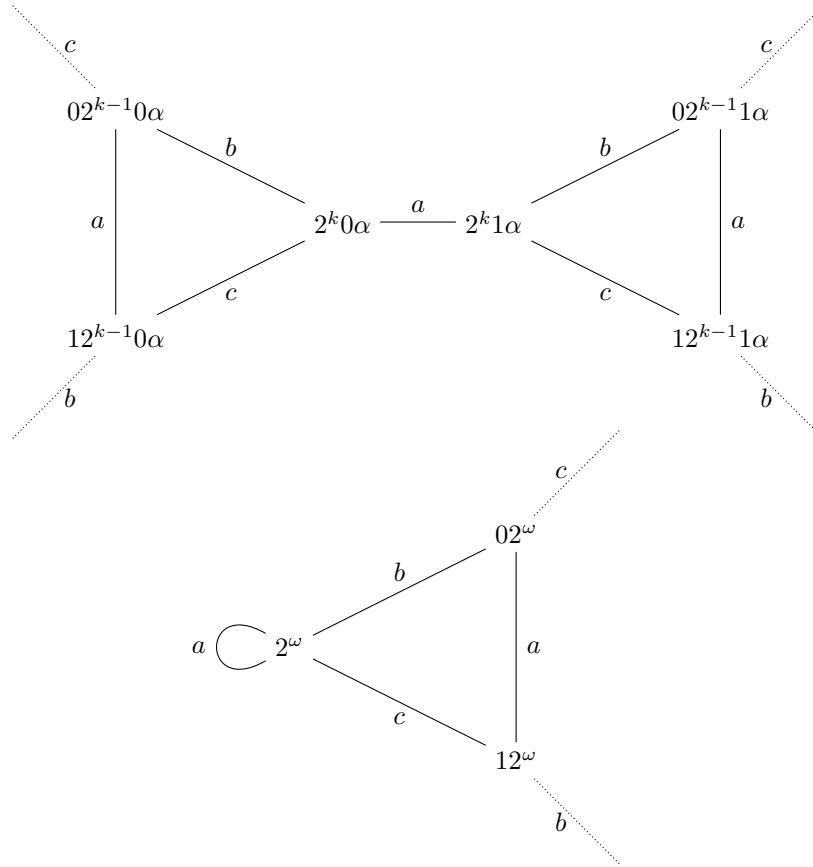


FIGURE 6. Part of the (finite) Schreier graphs in the Hanoi Towers group $H^{(3)}$ of $2^k 0\alpha$ (above) and of the (infinite) Schreier graph of 2^ω (below). See proof of Theorem 4.10.

Theorem 4.10. $F(\eta_n^i)$ converges to Υ_i for $i \in \{0, 1, 2\}$ as $n \rightarrow \infty$.

Proof. Let $(z_n^i)_{n \in \mathbb{N}}$ be a sequence of natural numbers such that z_n^i is the position of the first letter in η_n^i different from i . It comes from the structure of finite Schreier graphs of $H^{(3)}$ (see Fig. 6) that the balls of radius $z_n^i - 1$ in Υ_i rooted at i^ω , and in $\text{Sch}(\eta_n^i)$ rooted at η_n^i are isomorphic. Since z_n^i goes to infinity as n does, we have the assertion. \square

It can be shown that the infinite Schreier graphs of the Hanoi Tower group are all one ended, see [6]. Hence, since Υ_i is clearly two ended it follows that it is not an infinite Schreier graph of $H^{(3)}$. More precisely there is no $\xi \in \partial T = \Sigma^\omega$ such that the orbital Schreier graph $\text{Sch}(\xi)$ is isomorphic to Υ_i , even if the graph is considered non marked.

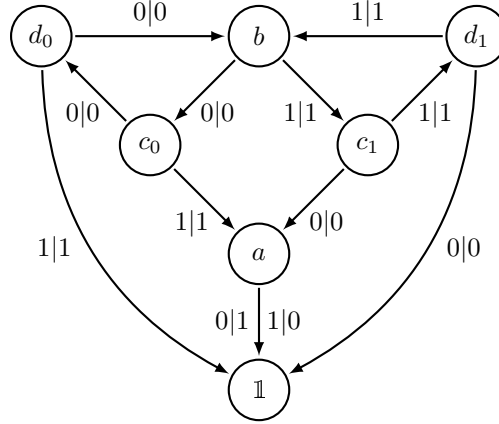


FIGURE 7. A twisted version of the Grigorchuk automaton.

Note that Grigorchuk automaton can be twisted in order to obtain a contracting automaton generating a fractal group with non countable κ , see Fig 7. However one can ask if there exist minimal automata where two singular points have isomorphic Shreier graphs, see Problem 2.

4.4. The bireversible case. In this section, we focus on the bireversible case. We start with the following alternative result which shows that, in the class of bireversible automata, the problem of finding examples with all continuous points in the boundary is equivalent to look for automata with all trivial stabilizers in the boundary.

Proposition 4.11. *Let \mathcal{M} be a bireversible automaton. For any $\xi \in \Sigma^\omega$, we have: $\xi \in \kappa \Leftrightarrow \text{Stab}_{\langle \mathcal{M} \rangle}(\xi) \neq \{1\}$.*

Proof. Let $g \in \text{Stab}_{\langle \mathcal{M} \rangle}(\xi)$. We have only two possibilities: either there exists $n \geq 1$ such that $g \cdot \xi[:n] = 1$, or for every $n \geq 1$ one has $g \cdot \xi[:n] \neq 1$. In the first case, since \mathcal{M} is bireversible we may apply Lemma 3.1, which implies $g = 1$. On the other hand, suppose $g \cdot \xi[:n] = g_n \in \langle \mathcal{M} \rangle \setminus \{1\}$. We have already remarked that $\text{St} : \partial T \rightarrow \text{Sub}(\langle \mathcal{M} \rangle)$ is not continuous at ξ if $\text{Stab}_{\langle \mathcal{M} \rangle}^0(\xi) \neq \text{Stab}_{\langle \mathcal{M} \rangle}(\xi)$. Since $g_n \neq 1$, for every n there exists $w_n \in \Sigma^*$ such that $g_n \circ w_n \neq w_n$. Let $\eta^{(n)} := \xi[:n]w_n\xi[n+|w_n|+1:] \in \Sigma^\omega$. Notice that $\eta^{(n)} \rightarrow \xi$, so that for every

neighborhood U of ξ there exists $k = k_U \geq 1$ such that $\eta^{(k)} \in U$. By hypothesis $g \in \text{Stab}_{\langle \mathcal{M} \rangle}(\xi)$ and $g \cdot \eta^{(k)}[:k] = g \cdot \xi[:k] = g_k$. The sequence $\eta^{(n)}$ converges to ξ since they have the same prefix of length n , so given U we find k such that $\eta^{(k)} \in U$. We have

$$\begin{aligned} g \circ \eta^{(k)} &= g \circ \xi[:k] ((g \cdot \eta[:k]) \circ w_k \xi[n + |w_n| + 1 :]) \\ &= \xi[:k] (g_k \circ w_k) ((g_k \cdot w_k) \circ \xi[n + |w_n| + 1 :]) \\ &\neq \eta^{(k)}. \end{aligned}$$

This implies that $\text{Stab}_{\langle \mathcal{M} \rangle}^0(x) \neq \text{Stab}_{\langle \mathcal{M} \rangle}(x)$, which means $x \in \kappa$. \square

We recall that the action of $\langle \mathcal{M} \rangle$ on Σ^ω is *essentially free* if

$$\mu(\{x \in \Sigma^\omega : \text{Stab}_{\langle \mathcal{M} \rangle}(x) \neq \{1\}\}) = 0,$$

where μ is the uniform measure on Σ^ω (see [15]). One may prove that groups generated by bireversible automata give rise to essentially free actions on Σ^ω [32]. Equivalently, this fact may be deduced by Proposition 4.11 and Lemma 4.2.

The next proposition characterizes in terms of Schreier graphs those automaton groups having all trivial stabilizers in the boundary, which by the previous proposition are those with $\kappa = \emptyset$.

Proposition 4.12. *Let \mathcal{M} be a bireversible automaton. The set κ of singular points is empty if and only if any two words in a same orbit cannot be cofinal.*

Proof. First, suppose that there exist $\xi \in \Sigma^\omega$ and $1 \neq g \in \langle \mathcal{M} \rangle$ such that $\eta := g \circ \xi \propto \xi$. Let us prove that there is an element of Σ^ω whose stabilizer is not trivial. Since $\eta \propto \xi$, then there exists $N > 0$ such that $\xi[N:] = \eta[N:]$. Since \mathcal{M} is bireversible, from Lemma 3.1 $g_k := g \cdot \xi[:k-1] \neq 1$ for any k . Since $g \circ \xi = \eta$, one gets

$$g_k \circ \xi[k:] = (g \cdot \xi[:k-1]) \circ \xi[k:] = \eta[k:].$$

In particular, for $k = N$ we get

$$g_N \circ \xi[N:] = g_N \circ \eta[N:] = \eta[N:] = \xi[N:].$$

This implies that $\xi' := \xi[N:]$ is stabilized by $g_N \neq 1$ and so it is a boundary point with a non-trivial stabilizer, so it is a singular point.

On the other hand, let $\langle \mathcal{M} \rangle$ admit a boundary point ξ with a non-trivial stabilizer. Then, there exists $1 \neq g \in \langle \mathcal{M} \rangle$ such that $g \cdot \xi = g$. Since $g \neq 1$ and \mathcal{M} is bireversible, there exist $v \in \Sigma^*$ and $f \in Q^*$, such that $f \circ v \neq v$ and $f \cdot v = g$. Hence $v\xi$ and $f \circ (v\xi)$ are two distinct cofinal words. \square

5. COMMUTING PAIRS AND DYNAMICS ON THE BOUNDARY

Theorem 4.4 from Section 4 states that the set κ of all singular points has measure zero. In this section we are interested in seeking for examples of automata with $\kappa = \emptyset$. Note that by Proposition 4.11, in the bireversible case this property is equivalent to have all trivial stabilizers in the boundary [16]. On the other hand, no example of such dynamics is known in this class (which seems to be the most difficult case), unless the generated group is finite. In this section we focus our attention on the class of reversible invertible automata.

5.1. Commuting pairs. Given an automaton $\mathcal{M} = (Q, \Sigma, \cdot, \circ)$, we say that $v \in \Sigma^*$, $u \in Q^*$ *commute* whenever

$$u \cdot v = u \text{ and } u \circ v = v ;$$

in this case we say that (u, v) is a *commuting pair*. The previous definition considers words $v \in \Sigma^*$, $u \in Q^*$. However, we may consider commuting pairs in $\mathcal{M} \sqcup \mathcal{M}^{-1} = (\tilde{Q}, \Sigma, \cdot, \circ)$. The importance of commuting pairs stems in the connection with the stabilizers of periodic points on the boundary. For example a classical way to prove that an automaton is not contracting has an interpretation in terms of commuting pairs: if there exists a commuting pair (u, v) such that u has infinite order then the automaton cannot be contracting [11].

We put

$$\text{Stab}_{\langle \mathcal{M} \rangle}^+(\xi) := \text{Stab}_{\langle \mathcal{M} \rangle}(\xi) = \pi(Q^*) \cap \text{Stab}_{\langle \mathcal{M} \rangle}(\xi)$$

as the set of “positive” stabilizers. We have the following proposition that clarifies the connection between commuting pairs and stabilizers.

Proposition 5.1. *Let \mathcal{M} be an invertible automaton. If $v \in \Sigma^*$, $u \in \tilde{Q}^*$ ($u \in Q^*$) commute, then $\pi(u) \in \text{Stab}_{\langle \mathcal{M} \rangle}(v^\omega)$ (respectively, $\text{Stab}_{\langle \mathcal{M} \rangle}^+(v^\omega)$).*

Conversely, for any $v \in \Sigma^$, $\pi(u) \in \tilde{Q}^*$ ($\pi(u) \in Q^*$) such that $\pi(u) \in \text{Stab}_{\langle \mathcal{M} \rangle}(v^\omega)$ (respectively, $u \in \text{Stab}_{\langle \mathcal{M} \rangle}^+(v^\omega)$) then there are integers $n, j \geq 1$ such that $u \cdot v^j, v^n$ commute. Moreover if \mathcal{M} is an RI-automaton then one can take $j = 0$, i.e. u, v^n commute.*

Proof. The first statement follows from $u \circ v^i = v^i$, $u \cdot v^i = u$, for all $i \geq 1$, i.e. $\pi(u) \in \text{Stab}_{\langle \mathcal{M} \rangle}(v^\omega)$ ($\pi(u) \in \text{Stab}_{\langle \mathcal{M} \rangle}^+(v^\omega)$ in case $u \in Q^*$). On the other hand, since the stateset is finite there exists k such that, $\forall j \geq k$, $u \cdot v^j$ is in the strongly connected component of $u \cdot v^k$. Then by strong connectivity there exist n and $j \geq k$ such that $(u \cdot v^j) \cdot v^n = u \cdot v^j$. Moreover $(u \cdot v^j) \circ v^n = v^n$, hence $(u \cdot v^j, v^n)$ is a commuting pair. In particular if \mathcal{M} is in addition reversible then every connected component is in fact strongly connected, hence one can choose $j = 0$. \square

The helix graph of $\mathcal{M} \sqcup \mathcal{M}^{-1}$ is denoted by $\tilde{\mathcal{H}}_{k,n}$. The following proposition gives a way to build all pairs of commuting pairs by looking at the “labels” of the cycles of the helix graphs.

Lemma 5.2. *Let \mathcal{M} be an invertible automaton. Let*

$$(u_0, v_0) \longrightarrow (u_1, v_1) \longrightarrow \cdots \longrightarrow (u_m, v_m) \longrightarrow (u_0, v_0)$$

be a cycle in the helix graph $\mathcal{H}_{k,n}$ ($\tilde{\mathcal{H}}_{k,n}$). Let $u = u_m \cdots u_0$ and $v = v_0 \cdots v_m$, then u, v commute.

Conversely, for any commuting pair $v \in \Sigma^$, $u \in Q^*$ ($u \in \tilde{Q}^*$) there is a helix graph $\mathcal{H}_{k,n}$ ($\tilde{\mathcal{H}}_{k,n}$) such that $(u, v) \longrightarrow (u, v)$.*

Furthermore, for any $u_0 \in Q^k$ ($u_0 \in \tilde{Q}^k$), $v_0 \in \Sigma^n$ there is a path

$$(u_0, v_0) \longrightarrow (u_1, v_1) \longrightarrow \cdots \longrightarrow (u_\ell, v_\ell) \longrightarrow \cdots \longrightarrow (u_\ell, v_\ell)$$

in the helix graph $\mathcal{H}_{k,n}$ ($\tilde{\mathcal{H}}_{k,n}$), for some $\ell \geq 0$.

Proof. From the definition of the helix graph we get:

$$(u_m \cdots u_0) \circ (v_0 \cdots v_m) = v_0 \cdots v_m, \quad (u_m \cdots u_0) \cdot (v_0 \cdots v_m) = u_m \cdots u_0$$

that is, u, v commute. On the other hand, if $u \cdot v = u$ and $u \circ v = v$, then the path $(u, v) \longrightarrow (u, v)$ is a cycle in $\mathcal{H}_{k,n}$ for $k = |u|$, $n = |v|$. The last statement is a consequence of the determinism of an automaton and the fact that any pair (u, v) has out-degree one, *i.e.*, for (u, v) there is exactly one pair (u', v') such that $(u, v) \longrightarrow (u', v')$ is an edge in the helix graph. \square

5.2. The reversible-invertible case. In this section we prove that those examples of *RI*-automata having all continuous points (for the map St) in the boundary (if any) are located in the class of bireversible automata. In other words an *RI*-automaton where the set of singular points is empty is necessarily bireversible. We also show a series of equivalences and connections with the property of having all trivial stabilizers in the boundary.

Examples of *RI*-automata with the aforementioned property are strictly related to a particular class of groups. Given a group G presented by $\langle X | R \rangle$, we say that G is *fully positive* whenever $R \subseteq X^*$, and for any $x \in X$ there is a word $u \in R$ such that x is a prefix of u . Note that the last property is equivalent to the fact that each $x^{-1} \in X^{-1}$ may be expressed as a positive element $u_x \in X^*$. These groups have the following alternative (apparently stronger) definition.

Proposition 5.3. *Let G be a group presented by $\langle X | R \rangle$, and let $\pi : \tilde{X}^* \rightarrow G$ be the natural map. The following are equivalent.*

- (i) G is fully positive;
- (ii) For any $u \in X^*$ there is $v \in X^*$ such that $\pi(uv) = \mathbb{1}$ in G ;
- (iii) For any $u \in X^*$ there is $v \in X^*$ such that $\pi(vu) = \mathbb{1}$ in G .

Proof. (ii) \Rightarrow (i). Follows by substituting every negative occurrence of a generator in a relator by a positive word.

(i) \Rightarrow (ii). Let us prove the statement by induction on the length $|u|$. The case $|u| = 1$ follows from the fact that G is fully positive. Therefore, suppose that the statement holds for $|u| < n$ and let us prove it for $|u| = n$. Consider any $u = au' \in X^n$ with $|u'| = n - 1$, for some $u' \in X^*$ and $a \in X$. By the induction hypothesis there is $v' \in X^*$ such that $\pi(u'v') = \mathbb{1}$. By the definition there is a defining relation $ah \in R$ starting with $a \in X$. Take $v = uv'h \in X^*$, then it is easy to check that $\pi(uv'h) = \pi(v) = \mathbb{1}$ holds. Equivalence (ii) \Leftrightarrow (iii) follows by conjugation. \square

Note that all torsion groups are fully positive. The following lemma is a direct consequence of Proposition 15 in [10].

Lemma 5.4. *Let $\mathcal{M} = (Q, \Sigma, \cdot, \circ)$ be an invertible automaton. Then, for any $\xi \in \Sigma^\omega$ and $\pi(u) \in \text{Stab}_{\langle \mathcal{M} \rangle}^+(\xi)$ ($\pi(u) \in \text{Stab}_{\langle \mathcal{M} \rangle}(\xi)$) there exist $i > j \geq 1$ such that $\pi(u \cdot \xi[: j]) \in \text{Stab}_{\langle \mathcal{M} \rangle}^+(\xi[j : i]^\omega)$ (respectively, $\pi(u \cdot \xi[: j]) \in \text{Stab}_{\langle \mathcal{M} \rangle}(\xi[j : i]^\omega)$).*

Proof. Let $\pi(u) \in \text{Stab}_{\langle \mathcal{M} \rangle}^+(\xi)$, then we have $u \circ \xi[: n] = \xi[: n]$ for all $n \geq 1$. Furthermore, by the finiteness of $Q^{|u|}$, and since $\{u \cdot \xi[: k]\}_{k \geq 0}$ is infinite, there are two indices $i > j \geq 1$ such that $u \cdot \xi[: i] = u \cdot \xi[: j]$. From which it follows that

$$(u \cdot \xi[: j]) \circ (\xi[j : i]^\omega) = \xi[j : i]^\omega$$

i.e., $\pi(u \cdot \xi[: j]) \in \text{Stab}_{\langle \mathcal{M} \rangle}^+(\xi[j : i]^\omega)$. The general case $\pi(u) \in \text{Stab}_{\langle \mathcal{M} \rangle}(\xi)$ is treated analogously considering $\mathcal{M} \sqcup \mathcal{M}^{-1}$ instead of \mathcal{M} . \square

From which we derive the two following consequences.

We have the following theorem.

Theorem 5.5. *Let $\mathcal{M} = (Q, \Sigma, \cdot, \circ)$ be an RI-automaton such that, for all $\xi \in \Sigma^\omega$, $\text{Stab}_{\langle \mathcal{M} \rangle}^+(\xi)$ contains only torsion elements. Then $\langle \mathcal{M} \rangle$ is fully positive.*

Proof. In particular, $\text{Stab}_{\langle \mathcal{M} \rangle}^+(v^\omega)$ contains only torsion elements for all $v \in \Sigma^*$. Take any arbitrary $u_0 \in Q^*$, we show that there is a relation having u_0 as a suffix, whence the statement follows by Proposition 5.3. By Lemma 5.2 there is path

$$(u_0, v_0) \longrightarrow \cdots \longrightarrow (u_\ell, v_\ell) \longrightarrow (u_{\ell+1}, v_{\ell+1}) \longrightarrow \cdots \longrightarrow (u_{\ell+k}, v_{\ell+k}) = (u_\ell, v_\ell)$$

in the helix graph $\mathcal{H}_{|u_0|, n}$ for some $v_0 \in \Sigma^n$, $\ell \geq 0$. Since we have a loop around the vertex (u_ℓ, v_ℓ) , by Lemma 5.2 $u = u_{\ell+1} \cdots u_{\ell+k}$ and $v = v_{\ell+1} \cdots v_{\ell+k}$ are a commuting pair. Proposition 5.1 implies $\pi(u) \in \text{Stab}_{\langle \mathcal{M} \rangle}^+(v^\omega)$, thus $\pi(u^m) = 1$ for some $m \geq 1$. Hence u_ℓ is the suffix of some relation. Since u_0 and u_ℓ belong to the same connected component, by using Lemma 3.1, we get a relation ending with u_0 . \square

Note that, in particular, if $\text{Stab}_{\langle \mathcal{M} \rangle}^+(\xi) = \{1\}$ for all $\xi \in \Sigma^\omega$, then $\langle \mathcal{M} \rangle$ is fully positive.

Here are two very similar results on stabilizers. One when stabilizers are trivial, the other when they contain only torsion elements.

Proposition 5.6. *Let $\mathcal{M} = (Q, \Sigma, \cdot, \circ)$ be an RI-automaton. The following are equivalent.*

- (i) $\text{Stab}_{\langle \mathcal{M} \rangle}^+(\xi) = \{1\}$ for all $\xi \in \Sigma^\omega$;
- (ii) $\text{Stab}_{\langle \mathcal{M} \rangle}^+(v^\omega) = \{1\}$ for all $v \in \Sigma^*$;
- (iii) $\text{Stab}_{\langle \mathcal{M} \rangle}(v^\omega) = \{1\}$ for all $v \in \Sigma^*$;
- (iv) $\text{Stab}_{\langle \mathcal{M} \rangle}(\xi) = \{1\}$ for all $\xi \in \Sigma^\omega$.

Proof. (i) \Leftrightarrow (ii). The implication is trivial. Let us look at the reciprocal implication: let $\pi(u) \in \text{Stab}_{\langle \mathcal{M} \rangle}^+(\xi)$. By Lemma 5.4, there are integers $i > j \geq 1$ such that $\pi(u \cdot \xi[:j]) \in \text{Stab}_{\langle \mathcal{M} \rangle}^+(\xi[j : i]^\omega)$, which is trivial by hypothesis. Hence $\pi(u \cdot \xi[:j]) = 1$, and so, by reversibility and Lemma 3.1, $\pi(u) = 1$.

(ii) \Leftrightarrow (iii). The converse (iii) \Rightarrow (ii) is trivial. Conversely, if (ii) holds, then by Theorem 5.5 we have that $\langle \mathcal{M} \rangle$ is fully positive. Therefore, for any $u \in \tilde{Q}$ we can construct a word $u^+ \in Q^*$ such that $\pi(u^+) = \pi(u)$. The result follows.

(iii) \Leftrightarrow (iv) follows from [10, Proposition 15]. \square

Proposition 5.7. *Let $\mathcal{M} = (Q, \Sigma, \cdot, \circ)$ be an RI-automaton. The following are equivalent.*

- (i) $\text{Stab}_{\langle \mathcal{M} \rangle}^+(\xi)$ is formed by torsion elements, for all $\xi \in \Sigma^\omega$;
- (ii) $\text{Stab}_{\langle \mathcal{M} \rangle}^+(v^\omega)$ is formed by torsion elements for all $v \in \Sigma^*$;
- (iii) $\text{Stab}_{\langle \mathcal{M} \rangle}(v^\omega)$ is a torsion group for all $v \in \Sigma^*$;

Furthermore, if \mathcal{M} is bireversible, following is also equivalent:

- (iv) $\text{Stab}_{\langle \mathcal{M} \rangle}(\xi)$ is a torsion group for all $\xi \in \Sigma^\omega$.

Proof. (i) \Leftrightarrow (ii). The implication is trivial. Let us look at the reciprocal implication: let $\pi(u) \in \text{Stab}_{\langle \mathcal{M} \rangle}^+(\xi)$. By Lemma 5.4, there are integers $i > j \geq 1$ such that $\pi(u \cdot \xi[:j]) \in \text{Stab}_{\langle \mathcal{M} \rangle}^+(\xi[j : i]^\omega)$, which contains only torsion elements by hypothesis.

Hence, there is an integer $\ell \geq 1$ such that $\pi((u \cdot \xi[:j])^\ell) = \mathbb{1}$. Further, since $\pi(u) \in \text{Stab}_{\langle \mathcal{M} \rangle}^+(\xi)$ the following equality:

$$(u \cdot \xi[:j])^\ell = u^\ell \cdot \xi[:j]$$

holds for any $j \geq 1$. Thus, by Lemma 3.1, we deduce $\pi(u) = \mathbb{1}$ and we get $\pi(u^\ell) = \pi((u \cdot \xi[:j])^\ell) = \mathbb{1}$, i.e., $\pi(u)$ is torsion.

(ii) \Leftrightarrow (iii). The proof is similar as the one in Proposition 5.6.

In the bireversible case, the implication (iv) \Rightarrow (iii) is trivial, while the converse is proven similarly to Proposition 5.6. Take $\pi(u) \in \text{Stab}_{\langle \mathcal{M} \rangle}(\xi)$. By Lemma 5.4, there are integers $i > j \geq 1$ such that $\pi(u \cdot \xi[:j]) \in \text{Stab}_{\langle \mathcal{M} \rangle}(\xi[j:i]^\omega)$. Since the group $\text{Stab}_{\langle \mathcal{M} \rangle}(\xi[j:i]^\omega)$ is torsion, then there is an integer $\ell \geq 1$ such that $\pi((u \cdot \xi[:j])^\ell) = \mathbb{1}$. Hence, by the bireversibility and Lemma 3.1, we get $\pi(u^\ell) = \pi(u^\ell \cdot \xi[:j]) = \pi((u \cdot \xi[:j])^\ell) = \mathbb{1}$. \square

In particular, note that the previous propositions imply the existence of a non-trivial “positive” stabilizer whenever the action on the boundary for an *RI*-automaton has at least one non-trivial stabilizer. Let $\mathcal{M} = (Q, \Sigma, \cdot, \circ)$ be an invertible automaton, and let $\langle \mathcal{M} \rangle = F_Q/N$. The set of “positive relations” of $\langle \mathcal{M} \rangle$ is

$$\mathcal{P}(\mathcal{M}) = Q^+ \cap \psi^{-1}(N)$$

where $\psi : \tilde{Q}^* \rightarrow F_Q$ is the canonical homomorphism. Note that $\mathcal{P}(\mathcal{M}) = \emptyset$ implies that $\langle \mathcal{M} \rangle^+$ is torsion-free and therefore infinite.

Lemma 5.8. *If $\mathcal{P}(\mathcal{M}) = \emptyset$, then $\text{Stab}_{\langle \mathcal{M} \rangle}^+(v^\omega) \neq \{\mathbb{1}\}$ for some $v \in \Sigma^*$.*

Proof. Let

$$(u_0, v_0) \longrightarrow \cdots \longrightarrow (u_\ell, v_\ell) \longrightarrow (u_0, v_0)$$

be a cycle in the helix graph $\mathcal{H}_{1,1}$. Then by Lemma 5.2 $(u_0 \dots u_\ell, v_0 \dots v_\ell)$ is a commuting pair. Hence $\pi(u_0 \dots u_\ell) \in \text{Stab}_{\langle \mathcal{M} \rangle}^+((v_0 \dots v_\ell)^\omega)$, and since $\mathcal{P}(\mathcal{M}) = \emptyset$, we conclude $\text{Stab}_{\langle \mathcal{M} \rangle}^+((v_0 \dots v_\ell)^\omega) \neq \{\mathbb{1}\}$. \square

The next theorem shows that either $\langle \mathcal{M} \rangle$ or $\langle \mathfrak{d}\mathcal{M} \rangle$ may have all trivial stabilizers in the boundary. We first recall the following proposition.

Proposition 5.9. [8, Corollary 5] *Let $\mathcal{M} = (Q, \Sigma, \cdot, \circ)$ be an *RI*-automaton with $\langle \mathcal{M} \rangle$ infinite. Then the index $[\langle \mathcal{M} \rangle : \text{Stab}_{\langle \mathcal{M} \rangle}(y^\omega)]$ is infinite for all $y \in \Sigma^*$, if and only if $\mathcal{P}(\mathfrak{d}\mathcal{M}) = \emptyset$.*

The following simple proposition is technical:

Proposition 5.10. *Let $\mathcal{M} = (Q, \Sigma, \cdot, \circ)$ be an *RI*-automaton with $\langle \mathcal{M} \rangle$ infinite. If $\text{Stab}_{\langle \mathcal{M} \rangle}^+(y^\omega) = \{\mathbb{1}\}$ for all $y \in \Sigma^*$, we have $\mathcal{P}(\mathfrak{d}\mathcal{M}) = \emptyset$.*

Proof. Suppose $\text{Stab}_{\langle \mathcal{M} \rangle}^+(y^\omega) = \{\mathbb{1}\}$ for all $y \in \Sigma^*$. Let us prove

$$[\langle \mathcal{M} \rangle : \text{Stab}_{\langle \mathcal{M} \rangle}(y^\omega)] = \infty$$

for all $y \in \Sigma^*$. Indeed, if $[\langle \mathcal{M} \rangle : \text{Stab}_{\langle \mathcal{M} \rangle}(y^\omega)] < \infty$ for some $y \in \Sigma^*$, then the following property holds:

$$(1) \quad \exists k \leq [\langle \mathcal{M} \rangle : \text{Stab}_{\langle \mathcal{M} \rangle}(y^\omega)] \text{ such that } \forall g \in \text{Stab}_{\langle \mathcal{M} \rangle}^+(y^\omega) : g^k = \mathbb{1}.$$

In particular, by Proposition 5.3 we get that $\langle \mathcal{M} \rangle$ is fully positive. We claim that every element of $\langle \mathcal{M} \rangle$ is torsion. Indeed, since $\langle \mathcal{M} \rangle$ is fully positive we have that

for any $q \in Q$ there is a $u_q \in Q^*$ such that $\pi(q^{-1}) = \pi(u_q)$. Now take any $w \in \tilde{Q}^*$, by substituting each negative occurrence $q^{-1} \in Q^{-1}$ appearing in w with u_q , we obtain a “positive” word $\hat{w} \in Q^*$ such that $\pi(\hat{w}) = \pi(w)$. Thus, by (1) we have:

$$\pi(w)^k = \pi(\hat{w})^k = \pi(\hat{w}^k) = \mathbb{1},$$

that is, $\langle \mathcal{M} \rangle$ is torsion. Being a residually finite group with uniformly bounded torsion, we deduce from [37, 38] that $\langle \mathcal{M} \rangle$ is finite, a contradiction. Hence, we obtain $[\langle \mathcal{M} \rangle : \text{Stab}_{\langle \mathcal{M} \rangle}(y^\omega)] = \infty$ for all $y \in \Sigma^*$, and so, by Proposition 5.9, $\mathcal{P}(\mathfrak{d}\mathcal{M}) = \emptyset$. In particular, by Corollary 5.8, $\mathcal{P}(\mathfrak{d}\mathcal{M}) = \emptyset$ implies $\text{Stab}_D^+(z^\omega) \neq \{\mathbb{1}\}$ for some $z \in Q^*$, hence the last statement holds. \square

The latter admits a partial converse in the case where there exists an aperiodic element in $\langle \mathcal{M} \rangle$:

Proposition 5.11. *Let \mathcal{M} be a RI Mealy automaton. If $\langle \mathcal{M} \rangle$ is not torsion and $\mathcal{P}(\mathfrak{d}\mathcal{M}) \neq \emptyset$ then*

$$\exists y \in \Sigma^*, \text{Stab}_{\langle \mathcal{M} \rangle}^+(y) \neq \{\mathbb{1}\}$$

Proof. Let $u \in \tilde{Q}^*$ such that $\pi(u)$ is aperiodic. Since $\mathcal{P}(\mathfrak{d}\mathcal{M})$ is not empty there exists y such that $\sigma(y) = \mathbb{1}$. Then in the helix graph $\tilde{\mathcal{H}}(\mathcal{M})_{|u|,|v|}$, there is a path

$$(u, y) \longrightarrow (u, y_1) \longrightarrow \cdots \longrightarrow (u, y_k) \longrightarrow (u, y_{k+1}) \longrightarrow \cdots \longrightarrow (u, y_\ell) \longrightarrow (u, y_k).$$

Hence u^{l-k} and $y_k \cdots y_\ell$ commutes. Then $\pi(u^{l-k}) \in \text{Stab}_{\langle \mathcal{M} \rangle}((y_k \cdots y_\ell)^\omega)$ and $\pi(u^{l-k}) \neq \mathbb{1}$. We conclude using Prop. 5.6. \square

Theorem 5.12. *Let $\mathcal{M} = (Q, \Sigma, \cdot, \circ)$ be an RI-automaton with $\langle \mathcal{M} \rangle$ infinite. We have either $\text{Stab}_{\langle \mathcal{M} \rangle}^+(y^\omega) \neq \{\mathbb{1}\}$ for some $y \in \Sigma^*$, or $\text{Stab}_{\langle \mathfrak{d}\mathcal{M} \rangle}^+(z^\omega) \neq \{\mathbb{1}\}$ for some $z \in Q^*$.*

Now, we focus on the possible consequences of being bireversible or not. We first recall the following proposition.

Proposition 5.13. [8, 14] *Let \mathcal{M} be an RI-automaton. If \mathcal{M} does not contain a bireversible connected component, then $\mathcal{P}(\mathcal{M}) = \emptyset$.*

From which we have the following proposition.

Proposition 5.14. *Let $\mathcal{M} = (Q, \Sigma, \cdot, \circ)$ be a non bireversible RI-automaton. Then $\text{Stab}_{\langle \mathcal{M} \rangle}^+(v^\omega) \neq \{\mathbb{1}\}$ for some $v \in \Sigma^*$. Furthermore, the set \mathcal{K} of singular points is not empty.*

Proof. The first statement follows by applying Proposition 5.13 and Corollary 5.8 to a non bireversible component of \mathcal{M} . Let us prove the last claim of the proposition. We show that St is actually not continuous at v^ω . Indeed, let $\pi(u) \in \text{Stab}_{\langle \mathcal{M} \rangle}^+(v^\omega)$ with $\pi(u) \neq \mathbb{1}$, for some $u \in Q^*$. If St were continuous at v^ω then, by Lemma 4.3, $\pi(u \cdot v^j) = \mathbb{1}$ for some $j \geq 1$. Thus, by Lemma 3.1, we get $\pi(u) = \mathbb{1}$, a contradiction. \square

As an immediate consequence of the previous proposition we obtain the following result.

Corollary 5.15. *If there exists an RI-automaton \mathcal{M} generating a group without singular points, then necessarily \mathcal{M} is bireversible.*

Let us have a look at the case of bireversible automata without singular points.

Proposition 5.16. *Let $\mathcal{M} = (Q, \Sigma, \cdot, \circ)$ be a bireversible automaton with $\langle \mathcal{M} \rangle$ infinite and no singular points. Then for any $u \in Q^*$ the following are equivalent:*

- (i) *the Schreier graph centered at u^ω is finite;*
- (ii) *there is an integer $\ell > 0$ such that $\pi(u^\ell) = \mathbb{1}$;*
- (iii) $\text{Stab}_{\langle \mathfrak{d}\mathcal{M} \rangle}^+(u^\omega) \neq \{\mathbb{1}\}$.

Proof. Recall that, for bireversible automata \mathcal{M} , having no singular points is equivalent to $\text{Stab}_{\langle \mathcal{M} \rangle}^+(v^\omega) = \{\mathbb{1}\}$ for all $v \in \Sigma^*$, by Propositions 5.7 and 4.11.

(i) \Leftrightarrow (ii) follows from [8, Theorem 6].

(iii) \Rightarrow (ii): Let $v \in \Sigma^*$ such that $\sigma(v) \in \text{Stab}_{\langle \mathfrak{d}\mathcal{M} \rangle}^+(v^\omega)$, where $\sigma : \tilde{\Sigma}^* \rightarrow \langle \mathfrak{d}\mathcal{M} \rangle$ is the natural map. By Proposition 5.1 there is an integer $\ell > 0$ such that v and u^ℓ commute, hence $\pi(u^\ell) \in \text{Stab}_{\langle \mathcal{M} \rangle}^+(v^\omega) = \{\mathbb{1}\}$, i.e., $\pi(u^\ell) = \mathbb{1}$.

(i) \Rightarrow (iii): If the Schreier graph centered at u^ω is finite, then, for any $v \in \Sigma^*$ we get $\sigma(v^k) \in \text{Stab}_{\langle \mathfrak{d}\mathcal{M} \rangle}(u^\omega)$ for $k = [\langle \mathfrak{d}\mathcal{M} \rangle : \text{Stab}_{\langle \mathfrak{d}\mathcal{M} \rangle}(u^\omega)]$. Furthermore, $\sigma(v^k) \neq \mathbb{1}$ since, by Proposition 5.10, $\mathcal{P}(\mathfrak{d}\mathcal{M}) = \emptyset$, whence $\text{Stab}_{\langle \mathfrak{d}\mathcal{M} \rangle}^+(u^\omega) \neq \{\mathbb{1}\}$. \square

The equivalence (ii) \Leftrightarrow (iii) of Prop. 5.16 links being torsion and having non-trivial stabilizers. We name it for future references in this paper:

$$(TS) \quad \forall u \in \Sigma^* \text{ we have } \text{Stab}_{\langle \mathcal{M} \rangle}^+(u^\omega) \neq \{\mathbb{1}\} \iff \sigma(u^\ell) = \mathbb{1}, \text{ for some } \ell \geq 1.$$

We recall that an *acyclic multidigraph* is a multidigraph without cycles. We have the following geometrical description in terms of some algebraic conditions.

Proposition 5.17. *Let \mathcal{M} be an RI-automaton. If $\mathcal{P}(\mathcal{M}) = \emptyset$ and (TS) then the orbital graphs $\Gamma(\langle \mathcal{M} \rangle, Q, \Sigma^\omega, v^\omega)$ of periodic points v^ω for $v \in \Sigma^*$, are either finite or acyclic multidigraphs.*

Proof. Suppose that both $\mathcal{P}(\mathcal{M}) = \emptyset$ and Condition (TS) hold. Let $u \in \Sigma^*$. By [8, Lemma 2] it follows that every vertex of $\Gamma(\langle \mathcal{M} \rangle, Q, \Sigma^\omega, u^\omega)$ is a periodic point h^ω for some $h \in \Sigma^*$. Assume that there exists $h \in \Sigma^*$ such that $\text{Stab}_{\langle \mathcal{M} \rangle}^+(h^\omega) \neq \{\mathbb{1}\}$, then by condition (TS) and [8, Theorem 6], $\Gamma(\langle \mathcal{M} \rangle, Q, \Sigma^\omega, h^\omega)$ is finite and so is $\Gamma(\langle \mathcal{M} \rangle, Q, \Sigma^\omega, u^\omega)$.

Otherwise, $\text{Stab}_{\langle \mathcal{M} \rangle}^+(p) = \{\mathbb{1}\}$ for each vertex p of $\Gamma(\langle \mathcal{M} \rangle, Q, \Sigma^\omega, u^\omega)$. Thus, by condition $\mathcal{P}(\mathcal{M}) = \emptyset$, we deduce that, for any vertex p of $\Gamma(\langle \mathcal{M} \rangle, Q, \Sigma^\omega, u^\omega)$ there is no cycle $p \xrightarrow{v} p$ for any $v \in Q^*$, i.e., $\Gamma(\langle \mathcal{M} \rangle, Q, \Sigma^\omega, u^\omega)$ is an acyclic multidigraph. \square

Note that the converse of the previous proposition holds if one assumes the existence of an infinite orbital graph rooted at some periodic point. Gathering Theorem 5.12 and the equivalence (TS), we obtain the following corollary, and so by Proposition 5.17 a description of the Schreier graphs of the dual in the case $\langle \mathcal{M} \rangle$ has no singular points.

Corollary 5.18. *Let \mathcal{M} be a bireversible automaton with $\langle \mathfrak{d}\mathcal{M} \rangle$ infinite and no singular points in the dual: $\forall \xi \in Q^\omega, \text{Stab}_{\langle \mathfrak{d}\mathcal{M} \rangle}(\xi) = \{\mathbb{1}\}$. Then (TS) holds.*

We can now obtain a lower bound on the growth of the Schreier graphs pointed at periodic points.

Proposition 5.19. *Let \mathcal{M} be a bireversible automaton with $\langle \mathcal{M} \rangle$ infinite and no singular points. Then for any $u \in Q^*$ with $\pi(u)$ aperiodic, we have*

$$\forall m \geq 1, \quad \left[\langle \mathfrak{d}\mathcal{M} \rangle : \text{Stab}_{\langle \mathfrak{d}\mathcal{M} \rangle}(u^m) \right] > \log_{|\Sigma|}(m).$$

Proof. Fix some $u \in Q^*$ with $\pi(u)$ aperiodic. We claim that for any $v \in \Sigma^*$, if $\sigma(v) \in \text{Stab}_{\langle \mathfrak{d}\mathcal{M} \rangle}^+(u^m) \setminus \{1\}$, then $m < |\Sigma|^{|v|}$. For $0 \leq i < m$, we have $\sigma(u^i o v) \in \text{Stab}_{\langle \mathfrak{d}\mathcal{M} \rangle}^+(u)$. By the reversibility of \mathcal{M} , there is an integer $k \leq |\Sigma|^{|v|}$ such that $u^k o v = v$. If $m \geq |\Sigma|^{|v|}$, then $k \leq m$ and $\sigma(v) \in \text{Stab}_{\langle \mathfrak{d}\mathcal{M} \rangle}^+(u^\omega)$, contradicting Corollary 5.18. Therefore, $m < |\Sigma|^{|v|}$. In particular, taking any $a \in \Sigma$, we get that there is an integer $j \leq [\langle \mathfrak{d}\mathcal{M} \rangle : \text{Stab}_{\langle \mathfrak{d}\mathcal{M} \rangle}(u^m)]$ such that $\sigma(a^j) \in \text{Stab}_{\langle \mathfrak{d}\mathcal{M} \rangle}(u^m)$, from which we obtain

$$\left[\langle \mathfrak{d}\mathcal{M} \rangle : \text{Stab}_{\langle \mathfrak{d}\mathcal{M} \rangle}(u^m) \right] \geq j > \log_{|\Sigma|}(m)$$

and this concludes the proof. \square

We obtain the following geometrical characterization.

Theorem 5.20. *Let \mathcal{M} be a bireversible automaton with $\langle \mathcal{M} \rangle$ infinite, with at least one aperiodic element. The following are equivalent.*

- (i) $\text{Stab}_{\langle \mathcal{M} \rangle}(\xi)$ is a torsion group, for all $\xi \in \Sigma^\omega$;
- (ii) $\mathcal{P}(\mathfrak{d}\mathcal{M}) = \emptyset$ and (TS) hold.
- (iii) the orbital graphs $\Gamma(\langle \mathfrak{d}\mathcal{M} \rangle, \Sigma, Q^\omega, v^\omega)$ of periodic points are either finite or acyclic multidigraphs.

Proof. (i) \Rightarrow (ii): From (i), the subgroup $\text{Stab}_{\langle \mathcal{M} \rangle}(y^\omega)$ is torsion for any $y \in \Sigma^*$. Let us prove $\mathcal{P}(\mathfrak{d}\mathcal{M}) = \emptyset$. Assume $[\langle \mathcal{M} \rangle : \text{Stab}_{\langle \mathcal{M} \rangle}(y^\omega)] < \infty$ for some $y \in \Sigma^*$. Then, the subgroup

$$N = \bigcap_{g \in \langle \mathcal{M} \rangle} g \text{Stab}_{\langle \mathcal{M} \rangle}(y^\omega) g^{-1}$$

is a finite index torsion normal subgroup of $\langle \mathcal{M} \rangle$. Thus, since $\langle \mathcal{M} \rangle/N$ is finite, the group $\langle \mathcal{M} \rangle$ is torsion, a contradiction. We deduce $[\langle \mathcal{M} \rangle : \text{Stab}_{\langle \mathcal{M} \rangle}(y^\omega)] = \infty$ for all $y \in \Sigma^*$, whence by Proposition 5.9, $\mathcal{P}(\mathfrak{d}\mathcal{M}) = \emptyset$. Now, to prove (TS) we essentially repeat the proof of the equivalence (iii) \Leftrightarrow (ii) of Proposition 5.16; the only point where the torsion hypothesis is used, is in the implication (iii) \Rightarrow (ii), while the other parts may be repeat “verbatim”.

(ii) \Rightarrow (i): By Proposition 5.7 it is enough to prove that $\text{Stab}_{\langle \mathcal{M} \rangle}^+(y^\omega)$, $y \in \Sigma^*$, are formed by torsion elements. Thus, let $\pi(u) \in \text{Stab}_{\langle \mathcal{M} \rangle}^+(y^\omega)$ for some $u \in Q^*$, $y \in \Sigma^*$. By Proposition 5.1, u, y^n is a commuting pair for some integer $n \geq 1$, and by the same proposition $\sigma(y^n) \in \text{Stab}_{\langle \mathfrak{d}\mathcal{M} \rangle}(u^\omega)$. Therefore, since $\mathcal{P}(\mathfrak{d}\mathcal{M}) = \emptyset$, we find $\text{Stab}_{\langle \mathfrak{d}\mathcal{M} \rangle}^+(u^\omega) \neq \{1\}$, whence $\pi(u^\ell) = 1$, by Proposition 5.16, i.e. $\pi(u)$ is torsion.

(ii) \Rightarrow (iii): Direct by Proposition 5.17.

(iii) \Rightarrow (ii): Orbital graphs $\Gamma(\langle \mathfrak{d}\mathcal{M} \rangle, \Sigma, Q^\omega, v^\omega)$, $v \in Q^*$ cannot be all finite. Indeed, if $v \in Q^*$, the connected components of its powers have sizes bounded by the size of $\Gamma(\langle \mathfrak{d}\mathcal{M} \rangle, \Sigma, Q^\omega, v^\omega)$. Thus, by [22, Proposition 7] $\pi(v)$ is of finite order. And therefore, $\langle \mathcal{M} \rangle$ is torsion, which contradicts the fact that $\langle \mathcal{M} \rangle$ has an aperiodic element. Hence there is an infinite (acyclic by hypothesis) orbital graph $\Gamma(\langle \mathfrak{d}\mathcal{M} \rangle, \Sigma, Q^\omega, h^\omega)$, for some $h \in Q^*$. Now, if $\mathcal{P}(\mathfrak{d}\mathcal{M}) \neq \emptyset$, there is a non-trivial cycle $h^\omega \rightarrow h^\omega$ in

the previous acyclic multidigraph, which is impossible.

Finally, let us show (TS): in fact, when the orbital graph is finite, then both sides of the equivalence are true, otherwise neither of the sides holds.

Let $u \in Q^*$. If $\Gamma(\langle \mathfrak{d}\mathcal{M} \rangle, \Sigma, Q^\omega, u^\omega)$ is finite, then clearly $\text{Stab}_{\langle \mathfrak{d}\mathcal{M} \rangle}^+(u^\omega) \neq \{1\}$ holds and, by the previous argument, $\pi(u)$ is of finite order. On the other hand, if $\Gamma(\langle \mathfrak{d}\mathcal{M} \rangle, \Sigma, Q^\omega, u^\omega)$ is infinite, then $\pi(u)$ is aperiodic according to [22, Proposition 7]. Further, by hypothesis, the infinite orbital graph $\Gamma(\langle \mathfrak{d}\mathcal{M} \rangle, \Sigma, Q^\omega, u^\omega)$ is acyclic, whence $\text{Stab}_{\langle \mathfrak{d}\mathcal{M} \rangle}^+(u^\omega) = \{1\}$, and this concludes the proof. \square

In this section we have constructed tools to find singular points, especially in the case of bireversible automata. It is still unknown whether there exists examples of infinite groups generated by RI automata without singular point. In particular we connected singular points with helix graphs and we proved that search can be narrow down to fully positive groups (Theorem 5.5). We also discuss the connection between the existence of non-trivial stabilizer of a group generated by a RI-automaton and the stabilizers of its dual (Theorem 5.12). This analysis shows that the possible existence of a RI automaton generating a group with all trivial stabilizers can be restricted to the class of bireversible. However evidences suggest that if G , generated by a RI automaton is infinite then it admits at least a non-trivial boundary. stabilizers.

6. DYNAMICS AND WANG TILINGS

There is an interesting connection between commuting pairs and Wang tilings observed by I. Bondarenko [5]. Given a Mealy automaton, one may associate a set of Wang tiles reflecting the action of the automaton on the stateset and the alphabet. The existence of periodic tilings corresponds to the notion of commuting words that generate elements of the stabilizers of infinite periodic words. Note that this problem – called the *domino problem* – is undecidable in general [4, 21]. In this section we prove that the domino problem is decidable for some family of tilesets linked to Mealy automata. On the other hand, we show that the problem of determining whether or not an automaton has commuting words on a restricted stateset is undecidable.

6.1. Wang tiles vs cross-diagrams. We recall that a Wang tile is a unit square tile with a color on each edge. Formally, it is a quadruple $t = (t_w, t_s, t_e, t_n) \in C^4$ where C is a finite set of colors (see Fig. 8 for a typical depiction of a Wang tile). A tileset is a finite set \mathcal{T} of Wang tiles, and for each $t \in \mathcal{T}$ and $d \in \{n, s, e, w\}$, we

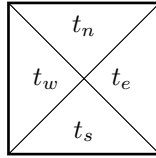


FIGURE 8. A Wang tile.

put t_d for the color of the edge in the d -side. Given a tileset \mathcal{T} , a tiling of the discrete plane is a map $f : \mathbb{Z}^2 \rightarrow \mathcal{T}$ that associates to each point in the discrete plane a tile from \mathcal{T} such that adjacent tiles share the same color on their common edge,

i.e., for any $(x, y) \in \mathbb{Z}^2$, $f(x, y)_e = f(x + 1, y)_w$ and $f(x, y)_n = f(x, y + 1)_s$. For a rectangle $[x, x'] \times [y, y'] \subseteq \mathbb{Z}^2$, and $d \in \{n, s, e, w\}$ we denote by $f([x, x'] \times [y, y'])_d$ the word in C^* labelling the d -side of the square. For instance for $d = w$ we have:

$$f([x, x'] \times [y, y'])_w = f(x, y)_w f(x, y + 1)_w \cdots f(x, y')_w.$$

This notion extends naturally for rectangles of the form $[x, \infty] \times [y, \infty] \subseteq \mathbb{Z}^2$.

A tiling f is *periodic* if there exists a periodicity vector $v \in \mathbb{Z}^2$ such that $f(t + v) = f(t)$ for $t \in \mathbb{Z}^2$. A tiling is *bi-periodic* if there are two linearly independent vectors v_1, v_2 for which f is periodic. It is a well known fact belonging to the folklore that a tiling admits a periodic tiling if and only if it admits a bi-periodic tiling if and only if it admits vertical and horizontal periods. In case $f : \mathbb{Z}^2 \rightarrow \mathcal{T}$ is a periodic tiling with a vertical period p_y and a horizontal period p_x , a fundamental domain of f is given by the square $[0, p_x - 1] \times [0, p_y - 1]$. Following [23] we say that the tiling \mathcal{T} is *cd-deterministic* with $(c, d) \in \{(e, n), (e, s), (w, n), (w, s)\}$ if each tile $t \in \mathcal{T}$ is uniquely determined by its pair (t_c, t_d) of colors. Whenever \mathcal{T} is *cd-deterministic* for each $(c, d) \in \{(e, n), (e, s), (w, n), (w, s)\}$, we say that \mathcal{T} is *4-way deterministic*.

There is a natural way to associate to a Mealy automaton $\mathcal{M} = (Q, \Sigma, \cdot, \circ)$ a tiling $\mathcal{T}(\mathcal{M})$: for each transition $q \xrightarrow{a|b} p$ we associate the Wang tile (q, a, p, b) with colors on $C = Q \sqcup \Sigma$, see Figure 9. This point of view is just a reformulation of the cross-diagram defined previously (and is completely different from the one in [12]). The following lemma links properties of the automaton \mathcal{M} with properties of the associated Wang tiling $\mathcal{T}(\mathcal{M})$.

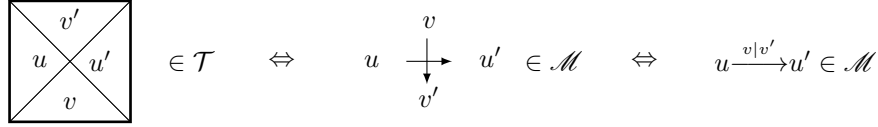


FIGURE 9. Tiling $\mathcal{T}(\mathcal{M})$ associated with an automaton \mathcal{M} .

Lemma 6.1. [5] *The tiling associated to a Mealy automaton is necessarily ws-deterministic. Furthermore, for a Mealy automaton \mathcal{M} we have the following.*

- $\mathcal{T}(\mathcal{M})$ is *es-deterministic* if and only if \mathcal{M} is *reversible*;
- $\mathcal{T}(\mathcal{M})$ is *wn-deterministic* if and only if \mathcal{M} is *invertible*;
- $\mathcal{T}(\mathcal{M})$ is *4-way deterministic* if and only if \mathcal{M} is *bireversible*.

There is a clear correspondence between tilings and cross diagrams, as illustrated in Figure 10. This easily extends to infinite words.

The following proposition links periodicity of a tiling and existence of a commuting pair for a Mealy automaton.

Proposition 6.2. *Let $\mathcal{M} = (Q, \Sigma, \cdot, \circ)$ be an automaton. The following are equivalent.*

- (i) $\mathcal{T}(\mathcal{M})$ admits a periodic tiling;
- (ii) \mathcal{M} admits a commuting pair: $v \in \Sigma^+$, $u \in Q^+$ with $u \circ v = v$, $u \cdot v = u$.

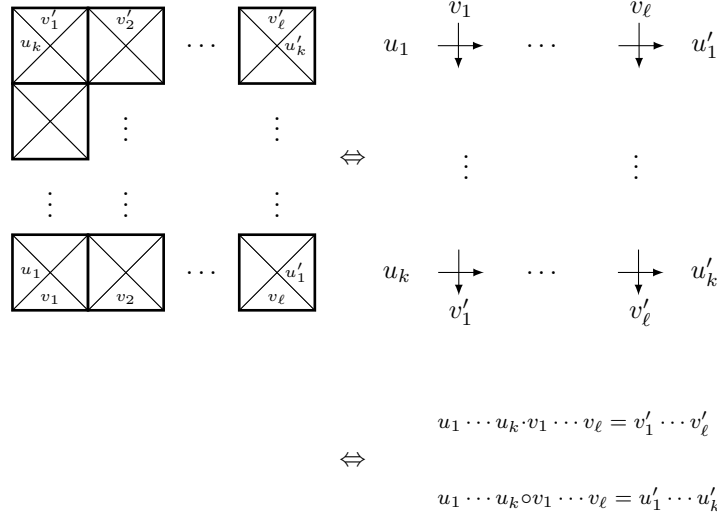


FIGURE 10. Correspondence between partial tilings, cross-diagrams and actions of the automaton.

Proof. If $\mathcal{T}(\mathcal{M})$ admits a periodic tiling, then there is a bi-periodic tiling $f : \mathbb{Z}^2 \rightarrow \mathcal{T}(\mathcal{M})$ with horizontal and vertical periods p_x, p_y , respectively. Therefore, the fundamental domain $[0, p_x - 1] \times [0, p_y - 1]$ pinpoints a rectangle in the tiling f with colors given by the string

$$f([0, p_x - 1] \times [0, p_y - 1])_n = f([0, p_x - 1] \times [0, p_y - 1])_s = v \in \Sigma^*$$

in both the south and north edge and

$$f([0, p_x - 1] \times [0, p_y - 1])_w = f([0, p_x - 1] \times [0, p_y - 1])_e = u \in Q^*$$

in both the west and east edge. Hence, by Figure 10: $u \circ v = v$, $u \cdot v = u$. On the other hand, if there is a commuting pair $u \in Q^*, v \in \Sigma^*$ such that $u \circ v = v$, $u \cdot v = u$ then by Figure 10 there is a partial tiling $f : [0, |v| - 1] \times [0, |u| - 1] \rightarrow \mathcal{T}(\mathcal{M})$ with

$$\begin{aligned}
f([0, |u| - 1] \times [0, |v| - 1])_n &= f([0, |u| - 1] \times [0, |v| - 1])_s = v, \\
f([0, |u| - 1] \times [0, |v| - 1])_w &= f([0, |u| - 1] \times [0, |v| - 1])_e = u.
\end{aligned}$$

This tiling extends naturally to a bi-periodic tiling $f : \mathbb{Z}^2 \rightarrow \mathcal{T}(\mathcal{M})$ of the whole discrete plane. \square

As a consequence we obtain a decidability result for the domino problem with particular sets of tilesets.

Starting from the tileset \mathcal{T} we construct a letter-to-letter transducer $\mathcal{M}(\mathcal{T}) = (Q, \Sigma, \circ)$ in a natural way as follows: Let $Q = \cup_{t \in \mathcal{T}} \{t_w, t_e\}$, $\Sigma = \cup_{t \in \mathcal{T}} \{t_s, t_n\}$, and transitions given by $t_w \xrightarrow{t_s | t_n} t_e \in \mathcal{M}(\mathcal{T})$ whenever $(t_w, t_s, t_e, t_n) \in \mathcal{T}$. Note that in general $\mathcal{M}(\mathcal{T})$ is not a Mealy automaton – we will deal with these cases in the next section. However when it is indeed a Mealy automaton the following holds:

Corollary 6.3. *Let \mathcal{T} be a Wang tileset. If $\mathcal{M}(\mathcal{T}) = (Q, \Sigma, \cdot, \circ)$ is a Mealy automaton then \mathcal{T} tiles the plane periodically, in an effective way.*

Proof. It is decidable if $\mathcal{M}(\mathcal{T})$ is a Mealy automaton in time $O(|Q||\Sigma|)$. Now construct the helix graph $\mathcal{H}_{1,1}$ (once again in time $O(|Q||\Sigma|)$). Since the stateset and the alphabet are finite, and each pair $(u, v) \in \mathcal{H}_{1,1}$ has exactly one successor, there exists a cycle

$$(u_0, v_0) \longrightarrow (u_1, v_1) \longrightarrow \cdots \longrightarrow (u_m, v_m) \longrightarrow (u_0, v_0)$$

in $\mathcal{H}_{1,1}$. Hence by Lemma 5.2 there is a commuting pair $(u, v) \in Q^+ \times \Sigma^+$, so by Proposition 6.2, \mathcal{T} tiles periodically the plane. \square

6.2. Commuting pairs on a restricted stateset. With Corollary 6.3, we have seen that each tileset $\mathcal{T}(\mathcal{M})$ associated with a Mealy automaton $\mathcal{M} = (Q, \Sigma, \cdot, \circ)$ admits a periodic tiling. However (u, v) could commute because the word u acts like the identity, *i.e.* $\pi(u) = 1$; so this commuting pair will not help us to find singular points, and we may want to avoid it.

In this spirit we now consider *commuting pair on a restricted stateset*, that is commuting pair in $Q'^* \times \Sigma^*$, with $Q' \subsetneq Q$.

We now consider an automaton $\mathcal{M} = (Q, \Sigma, \cdot, \circ)$ and a subset $Q' \subsetneq Q$. We consider the *restricted tileset* $\mathcal{T}(\mathcal{M}, Q')$ formed by the tiles of $\mathcal{T}(\mathcal{M})$ whose colors are in the set $Q' \cup \Sigma$. Note that $\mathcal{T}(\mathcal{M}, Q')$ is the tileset associated with the (partial) automaton obtained from \mathcal{M} by erasing all the transitions to or from states in $Q \setminus Q'$. In particular when the automaton has a sink-state¹ e , we define *non-elementary commuting pairs*, that are commuting pairs restricted to the stateset $Q \setminus \{e\}$, and the *non-elementary tileset* $\mathcal{T}(\mathcal{M}, Q \setminus \{e\})$ will be denoted by $\overline{\mathcal{T}}(\mathcal{M})$. We have the following proposition.

Proposition 6.4. *Let \mathcal{M} be a Mealy automaton with a sink-state. Then \mathcal{M} admits a non-elementary commuting pair if and only if $\overline{\mathcal{T}}(\mathcal{M})$ has a periodic tiling.*

Proof. The proof is similar to the one presented in Proposition 6.2. Indeed if e were to appear in the cross diagramm, then since it is a sink it will stay present every step after, preventing the tiling to be periodic: $\forall u \in Q^*eQ^*, \forall v \in \Sigma^*, u \cdot v \in Q^*eQ^*$. \square

This raises the problem of finding restricted commuting pairs, since we can no longer apply Corollary 6.3. We obtain the following decision problem:

RESTRICTED COMMUTING PAIRS:

- **Input:** $\mathcal{M} = (Q, \Sigma, \cdot, \circ)$, $Q' \subsetneq Q$.
- **Output:** Does \mathcal{M} have commuting pairs restricted to the stateset Q' ?

It turns out that this problem is undecidable. To prove this we are going to reduce a known undecidable problem to it, namely the existence of a periodic tiling for a 4-way deterministic tileset (see [23]).

For the reduction from the periodic 4-way deterministic problem to our problem we show that, given a 4-way deterministic tileset, a periodic tiling can be determined from a non-trivial commuting pair of some Mealy automaton.

¹One can recognize trivial states in linear time (note that we can also find trivial components through *minimization*, with cost $O(|\Sigma||Q|\log|Q|)$, see [3]). In the following we assume that the automaton has at most one sink-state, which we denote e .

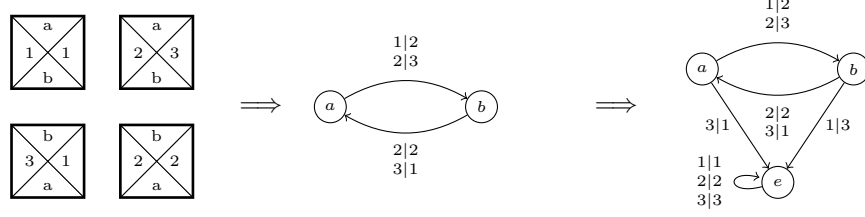


FIGURE 11. Construction of an invertible Mealy automaton from a 4-way deterministic tileset.

Let \mathcal{T} be a 4-way deterministic tileset. We build $\mathcal{M}(\mathcal{T})$ as in Corollary 6.3. This transducer is not in general complete, however, we may add an extra sink-state e and extra transitions to this sink-state in order to make the automaton complete and invertible. We call $\mathcal{M}_S(\mathcal{T})$ the invertible automaton associated with \mathcal{T} . Since the extra transitions go to the sink-state it is clear that $\mathcal{T} = \overline{\mathcal{T}}(\mathcal{M}_S(\mathcal{T}))$. We are now in position to prove the following result.

Theorem 6.5. *The problem RESTRICTED COMMUTING PAIRS is undecidable.*

Proof. We prove that a particular instance of RESTRICTED COMMUTING PAIRS is undecidable:

NON-ELEMENTARY COMMUTING PAIRS:

- **Input:** \mathcal{M} a Mealy automaton with a sink-state.
- **Output:** Does \mathcal{M} have non-elementary commuting pairs?

By [23] checking whether a 4-way deterministic tileset admits a periodic tiling is undecidable. By Proposition 6.4 checking whether \mathcal{T} admits a periodic tiling is equivalent to check whether $\mathcal{M}_S(\mathcal{T})$ has a non-elementary commuting pair, hence NON-ELEMENTARY COMMUTING PAIRS is undecidable, and so is RESTRICTED COMMUTING PAIRS. \square

Note that in this context we deal with inverses of Q , and it is not difficult to check that reduced words $u \in \widetilde{Q}^*$ may generate non-elementary commuting pair. Hence, in this context a non-elementary pair of commuting words is a pair of words $u \in \widetilde{Q \setminus \{e\}}^*$, $v \in \Sigma^*$ such that u is reduced and u, v commute.

Proposition 6.6. *With the above notation. Let $\mathcal{M} \in \mathcal{S}_a$. The following are equivalent.*

- (i) v^ω is a singular point, $v \in \Sigma^*$;
- (ii) there is a non-elementary commuting pair $u \in \widetilde{Q \setminus \{e\}}^*$, $v \in \Sigma^*$ with $\pi(u) \neq 1$.

Proof. (i) \Rightarrow (ii) If $v^\omega \in \kappa$ then the map St is not continuous at v^ω if and only if $\text{Stab}_{\langle \mathcal{M} \rangle}(v^\omega) \neq \text{Stab}_{\langle \mathcal{M} \rangle}^0(v^\omega)$. This implies that there exists $u \in \widetilde{Q \setminus \{e\}}^*$ satisfying $\pi(u) \in \text{Stab}_{\langle \mathcal{M} \rangle}(v^\omega) \setminus \text{Stab}_{\langle \mathcal{M} \rangle}^0(v^\omega)$. Note that $\pi(u \cdot v^k) \neq 1$ for any $k \geq 0$ (from Lemma 4.2). Since $|u|$ is finite, there exist $n > m > 0$ such that $u \cdot v^m = u \cdot v^n$. Then $u_m := u \cdot v^m$ and v^{n-m} form a non-elementary commuting pair with $u_m \in \widetilde{Q \setminus \{e\}}^*$. (ii) \Rightarrow (i) Let $u \in \widetilde{Q \setminus \{e\}}^*$, $v \in \Sigma^*$ be a non-elementary commuting pair. Since

$\pi(u) \neq 1$, there is a word $w \in \Sigma^*$ such that $uow \neq w$. Consider the sequence $w_n := v^n w v^\omega$ and proceed as in the proof of Lemma 4.3 to show that $\pi(u) \in \text{Stab}_{\langle \mathcal{M} \rangle}^0(v^\omega) \setminus \text{Stab}_{\langle \mathcal{M} \rangle}(v^\omega)$. \square

Remark 6.7. Note that Proposition 4.11 implies that Proposition 6.6 holds also for bireversible automata.

As we have already noted, the tileset $\mathcal{T}(\mathcal{M})$ associated with a Mealy automaton \mathcal{M} always admits a periodic tiling, however, in case we consider automata from \mathcal{S}_a , it is interesting, and useful for the sequel, to understand when the non-elementary tileset of $\overline{\mathcal{T}}(\mathcal{M})$ tiles the discrete plane. We need first the following lemma regarding inverse X -digraphs, that are digraph Γ with edges labelled by element of X , such that if $p \xrightarrow{a} q$ is an edge of Γ , so is $q \xrightarrow{a^{-1}} p$ (involutive), for any vertex $v \in \Gamma$ and any label $x \in \tilde{X}$, there is exactly one edge starting from v and labelled with x . In inverse digraph the natural quasimetric d is symmetric and so it is a distance.

Note that an inverse X -digraph has in- and out-degree $|\tilde{X}|$.

Lemma 6.8. Let Γ be an infinite connected inverse X -digraph, $|X| < \infty$, and let v be a vertex of Γ . Then there is a right-infinite word $\theta \in \tilde{X}^\omega$ such that

$$\lim_{i \rightarrow \infty} d(v, v_i) = \infty,$$

where v_i is defined by $v \xrightarrow{\theta[1:i]} v_i$.

Proof. Since Γ is an inverse digraph it is symmetric. Consider $\overline{\Gamma}$ the non-directed graph obtained by gluing edges $p \xrightarrow{a} q$ and $q \xrightarrow{a^{-1}} p$ in Γ . Note that the distances in Γ and in $\overline{\Gamma}$ are the same and we will denote both of them by $d(\cdot, \cdot)$. Since $\overline{\Gamma}$ has finite degree, is infinite and connected, we can apply König's lemma that claims that $\overline{\Gamma}$ contains an infinitely long simple path (that is, a path with no repeated vertices). Let $\nu = v_1 \cdots v_i \cdots$ denote such a path and θ the associated infinite word. Since $\overline{\Gamma}$ is connected there exists a path from any vertex v to v_1 : let $k = d(v, v_1)$. Since the graph $\overline{\Gamma}$ has bounded degree, the ball $B_k(v)$ of radius k centred in v is finite and since ν is simple there is an index i_k for which $v_j \notin B_k(v)$ for $j \geq i_k$. The same holds for arbitrary large k , which concludes the proof. \square

Consider the map λ from the set of edges of $\text{Sch}(\xi)$ to the integers, that associate to each edge $f = \eta \xrightarrow{q} \eta'$ the integer

$$\lambda(f) = \begin{cases} \min\{n : q \cdot \eta[1:n] = e\} & \text{if } \exists m : q \cdot \eta[1:m] = e \\ \infty & \text{otherwise.} \end{cases}$$

Let $d(\eta, \eta')$ denote the usual metric on $\text{Sch}(\xi)$. Since $\text{Sch}(\xi)$ is a regular digraph with finite out-degree \tilde{Q} for each integer n we consider the set of edges that admit at least one vertex inside the ball of radius n centered at ξ :

$$B_\xi^{\leftarrow}(n) = \{\eta \xrightarrow{q} \eta' : d(\xi, \eta) \leq n \vee d(\xi, \eta') \leq n\}.$$

Note that $B_\xi^{\leftarrow}(n)$ is clearly finite and we may define

$$\psi_\xi(n) = \max \{ \lambda(f) : f \in B_\xi^{\leftarrow}(n) \wedge \lambda(f) < \infty \}.$$

We remark that $\{ \lambda(f) : f \in B_\xi^{\leftarrow}(n) \wedge \lambda(f) < \infty \}$ is non-empty in view of the existence of the trivial edges of type $\eta \xrightarrow{e} \eta$. Thus, ψ_ξ is a monotonically increasing

function. We have thus two cases: either $\lim_{n \rightarrow \infty} \psi_\xi(n) = \infty$, or $\lim_{n \rightarrow \infty} \psi_\xi(n) = \ell$ for some $\ell \geq 1$. The following proposition provides sufficient conditions on the dynamics of the boundary for the non-elementary tileset to tile the discrete plane.

Proposition 6.9. *Let $\mathcal{M} = (Q, \Sigma, \cdot, \circ) \in \mathcal{S}_a$. If there is a point $\xi \in \Sigma^\omega$ whose Schreier graph $\text{Sch}(\xi)$ is infinite and such that $\lim_{n \rightarrow \infty} \psi_\xi(n) = \ell$ for some integer $\ell \geq 1$, then $\overline{\mathcal{T}}(\mathcal{M})$ tiles the discrete plane. Conversely, if $\overline{\mathcal{T}}(\mathcal{M})$ admits just aperiodic tilings, then there is an infinite Schreier graph $\text{Sch}(\xi)$ for some $\xi \in \Sigma^\omega$.*

Proof. It belongs to the folklore that a tileset tiles the whole discrete plane if and only if it tiles the first quadrant of the discrete plane (see for instance [28]). Let us prove that there is a tiling $f : [1, \infty] \times [1, \infty] \rightarrow \overline{\mathcal{T}}(\mathcal{M})$ of the first quadrant. First there is a tiling $h : [1, \infty] \times [1, \infty] \rightarrow \mathcal{T}(\mathcal{M})$ of the first quadrant such that, putting $\eta = h([1, \infty] \times [1, \infty])_s$ and $\sigma = h([1, \infty] \times [1, \infty])_w$, we have:

- (1) σ is not cofinal with e^ω , and
- (2) if $\sigma[j] \neq e$, for some $j \geq 1$, then $\sigma[j] \cdot (\sigma[:j-1] \circ \eta[:i]) \neq e$ for all $i \geq 1$.

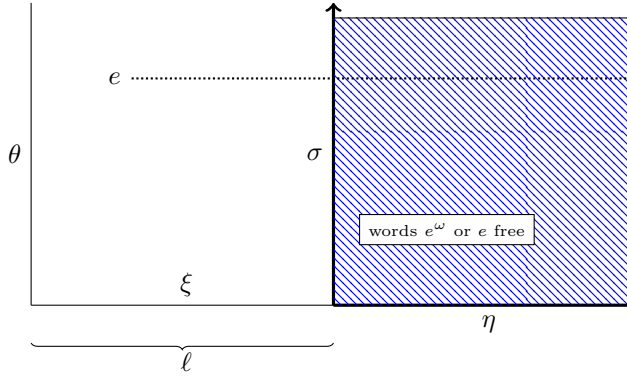


FIGURE 12. Construction of a non-elementary tiling when $\lim_{n \rightarrow \infty} \psi_\xi(n) = \ell$.

Intuitively, the last property means that if the west edge of the leftmost tile of a band is not colored by e , then also none of the west and east edges of all in this band are colored by e . Indeed, since e is a sink, if the west edge of the leftmost tile of a band is colored by e , then clearly all the west and east edges of the all the tiles in this band are colored by e as well, see Fig. 12. Let us first find a tiling fulfilling Condition 2: If there exists an element $q \in Q$ of infinite order, take q^ω as σ and some word of infinite orbit under the action of q as η . Otherwise, since $\text{Sch}(\xi)$ is infinite we may find, by Lemma 6.8, a right-infinite path starting at ξ and labelled by some right-infinite word $\theta \in \tilde{Q}^\omega$ such that $\lim_{i \rightarrow \infty} d(\xi, \theta[:i] \circ \xi) = \infty$. Since every element of Q has finite order we may remplace each negative occurrence of a letter by the suitable power of this letter, hence without loss of generality we can suppose $\theta \in Q^\omega$. By Figure 10 consider the tiling $h : [1, \infty] \times [1, \infty] \rightarrow \mathcal{T}(\mathcal{M})$ associated with ξ, θ , i.e., such that

$$\xi = h([1, \infty] \times [1, \infty])_s, \quad \theta = h([1, \infty] \times [1, \infty])_w.$$

Since $\lim_{n \rightarrow \infty} \psi_\xi(n) = \ell$, then for any vertex ξ' of $\text{Sch}(\xi)$ and $q \in Q$ the following property:

$$(2) \quad \text{if } q \cdot \xi'[:i] \neq e \text{ for all } 1 \leq i \leq \ell, \text{ then } q \cdot \xi'[:i] \neq e \text{ for all } i \geq \ell$$

holds. Since e is a sink-state, Equation (2) can be written equivalently:

$$(2) \quad \text{if } q \cdot \xi'[:\ell] \neq e, \quad \text{then } q \cdot \xi'[:i] \neq e \text{ for all } i \geq \ell.$$

Let us restrict the tiling h to the quadrant $[\ell, \infty] \times [1, \infty]$. Take

$$\begin{aligned} \eta &= h([\ell, \infty] \times [1, \infty])_s = \xi[\ell:], \\ \sigma &= h([\ell, \infty] \times [1, \infty])_w = \theta \cdot \xi[:\ell - 1]. \end{aligned}$$

Property 2 is obtained from Equation (2).

Let us prove now Property 1. Two elements $\xi', \xi'' \in \Sigma^\omega$ are m -cofinal if $\xi'[m:] = \xi''[m:]$. Note that for fixed $\xi' \in \Sigma^\omega$ and $m \geq 1$, the number of vertices ξ'' that are m -cofinal with ξ' is finite. Suppose that σ is cofinal with e^ω : $\sigma = \sigma[:k]e^\omega$ for some $k \geq 1$. Hence the sequence $(\sigma[:i] \circ \eta)_i$ is ultimately constant. But $\sigma[:i] \circ \eta$ is the suffix of $\theta[:i] \circ \xi$, whence the set $\{\theta[:k+j] \circ \xi : j \geq 1\}$ formed ℓ -cofinal vertices is finite. Which contradicts $\lim_{i \rightarrow \infty} d(\xi, \theta[:i] \circ \xi) = \infty$, and σ cannot be cofinite with e^ω . Whence we have obtained a tiling of the first quadrant with σ and η as west and south borders respectively, and such that, for all i , either $\sigma[i] \cdot \eta = e^\omega$ or $\sigma[i] \cdot \eta$ contains no e . We now obtain a tiling in $\overline{\mathcal{T}}(\mathcal{M})$ by deleting those line e^ω .

Let us prove the converse in our proposition. Let

$$\xi = f([1, \infty] \times [1, \infty])_s, \quad \theta = f([1, \infty] \times [1, \infty])_w$$

where $f : \mathbb{Z} \times \mathbb{Z} \rightarrow \overline{\mathcal{T}}(\mathcal{M})$ is an aperiodic tiling. Since f is aperiodic, we get $\theta[:i] \circ \xi \neq \theta[:j] \circ \xi$ for $i \neq j$ (since periodicity in one direction is equivalent to periodicity in both directions). Therefore, $\{\theta[:i] \circ \xi : i \geq 0\}$ is infinite and $\text{Sch}(\xi)$ as well. □

We recall that an automaton $\mathcal{M} = (Q, \Sigma, \cdot)$ is *synchronizing* whenever there is a word $w \in \Sigma^*$ such that $q \cdot w = p \cdot w$ for any $q, p \in Q$. A Mealy automaton $\mathcal{M} = (Q, \Sigma, \cdot, \circ)$ is synchronizing whenever (Q, Σ, \cdot) is synchronizing. The set of synchronizing words is denoted by $\text{Syn}(\mathcal{M})$. Note that an automaton in \mathcal{S}_a is always synchronizing. For further details on synchronizing automata and some connections with automaton groups see [9]. The next proposition characterizes automata in \mathcal{S}_a whose set of reduced tiles does not tile the discrete plane in terms of a stability property regarding synchronization.

Consider the prefix-closed language

$$\text{NSyn}_Q(v) = \{u \in Q^{<|v|-1} \mid \forall j \leq |u|, u[:j] \circ v \notin \text{Syn}(\mathcal{M})\}.$$

and the property **Maximal non Synchronizing** : there exists an integer m such that, for all $v \in \Sigma^m$, any u maximum in $\text{NSyn}_Q(v)$ for the prefix relation satisfies $uq \circ v \in \text{Syn}(\mathcal{M})$, for all $q \in Q$.

Proposition 6.10. *Let $\mathcal{M} = (Q, \Sigma, \cdot, \circ) \in \mathcal{S}_a$. Then $\overline{\mathcal{T}}(\mathcal{M})$ does not tile the discrete plane if and only if property **Maximal non Synchronizing** holds for \mathcal{M} .*

Proof. Suppose that $\overline{\mathcal{T}}(\mathcal{M})$ does not tile the discrete plane. Hence there exists an integer m such that $\overline{\mathcal{T}}(\mathcal{M})$ cannot tile a square of size m .

Let $v \in \Sigma^m$ and $u \in \text{NSyn}_Q(v)$. First suppose $|u| < m - 2$: if for some $q \in Q$ we have $uqov \notin \text{Syn}(\mathcal{M})$ then $uq \in \text{NSyn}_Q(v)$, hence u cannot be maximal. Now suppose $|u| = m - 2$: if $uqov \notin \text{Syn}(\mathcal{M})$, for some $q \in Q$, then there is a $q' \in Q$ such that $q' \cdot (uqov)[j] \neq e$ for all $j \leq m$. Hence, by Figure 10 there is a tiling of the square of size m associated with the two words v, uqq' , a contradiction.

Conversely, if $\overline{\mathcal{T}}(\mathcal{M})$ tiles a square $f : [1, m] \times [1, m] \rightarrow \overline{\mathcal{T}}(\mathcal{M})$, then if we put

$$h = f([1, m] \times [1, m])_s, \quad v = f([1, m] \times [1, m])_w$$

we get that $v[j] \circ h \notin \text{Syn}(\mathcal{M})$ for all $j \leq m - 1$. In particular, $u = v[m - 2]$ is maximal in $\text{NSyn}_Q(v)$. However, $(uv[m - 1]) \circ h = v[m - 1] \circ h \notin \text{Syn}(\mathcal{M})$, hence the condition of the statement is not satisfied. \square

6.3. Aperiodic tilings and singular points. In Subsection 5.2 we have seen that RI -automata, with empty set of singular points are necessarily bireversible with all trivial stabilizers in the boundary (see Corollary 5.15 and Proposition 4.11). Furthermore, Corollary 5.8 implies that if a bireversible automaton \mathcal{M} has no positive relations, then it must have a non-trivial stabilizer in the boundary. In particular, for the class of bireversible automata it is not possible to have simultaneously no positive relations and no singular points.

This fact no longer holds in the class \mathcal{S}_a : there exist automata in \mathcal{S}_a without positive relations and with no singular points. However, we need some precaution in defining the set of “positive relations” for an automaton $\mathcal{M} \in \mathcal{S}_a$. Indeed, for such an automaton, the set $\mathcal{P}(\mathcal{M})$ is always non-empty since the sink-state e acts like the identity.

In order to characterize tilings without non-trivial commuting pairs, we need to define a special set of relations that are in some sense non elementary. Such relations intuitively correspond to words with the property that some of their restrictions do not become the trivial word. For $w \in Q^*$, $|w|_e$ denotes the number of occurrences of the letter e in w . The set $\mathcal{E}(\mathcal{M})$ of the *non-elementary relations* is defined by:

$$\mathcal{E}(\mathcal{M}) = \{u \in (Q \setminus \{e\})^* : \pi(u) = \mathbb{1}, \exists v \in \Sigma^*, |u \cdot v^n|_e < |u|, \forall n \geq 1\}.$$

By a compactness argument note that the complement of the non-elementary relations, the set of *elementary relations*, may be described as the set of words $u \in (Q \setminus \{e\})^*$ with $\pi(u) = \mathbb{1}$, such that there exists some $n \geq 1$ for which $u \cdot v = e^{|v|}$ for every $v \in \Sigma^{\geq n}$. Geometrically, automata in the class \mathcal{S}_a with no singular points and whose eventual relations are elementary relations, possess helix graphs with a particular shape, as proved in Proposition 6.11. We say that the helix graph $\mathcal{H}_{k,n}$ of an automaton $\mathcal{M} \in \mathcal{S}_a$ is *singular* whenever each connected component of $\mathcal{H}_{k,n}$ has a unique cycle which is necessarily of the form $(e^k, v) \longrightarrow (e^k, v)$ for some $v \in \Sigma^n$.

Proposition 6.11. *Let $\mathcal{M} \in \mathcal{S}_a$. The following are equivalent:*

- (i) *for any $\xi \in \Sigma^\omega$, $g \in \text{Stab}_{\mathcal{M}}^+(\xi)$, there is $n \geq 1$ with $g \cdot \xi[n] = \mathbb{1}$ and $\mathcal{E}(\mathcal{M}) = \emptyset$;*
- (ii) *for any $v \in \Sigma^*$, $g \in \text{Stab}_{\mathcal{M}}^+(v^\omega)$, there is $n \geq 1$ with $g \cdot v^n = \mathbb{1}$ and $\mathcal{E}(\mathcal{M}) = \emptyset$;*
- (iii) *for any $k, n \geq 1$, the helix graph $\mathcal{H}_{k,n}$ is singular;*
- (iv) *there is no non-elementary pair of commuting words.*

Proof. (i) \Rightarrow (ii). Trivial.

(ii) \Rightarrow (iii). Suppose that $\mathcal{H}_{k,n}$ is not singular for some $k, n \geq 1$. Therefore, by Lemma 5.2 there is a commuting pair $u \in Q^* \setminus \{e\}^*, v \in \Sigma^*$. As e acts like the identity, by erasing the (potential) occurrences of e in u , we obtain a word $u' \in (Q \setminus \{e\})^*$ such that u', v commutes. Hence Lemma 5.2 implies $\pi(u') \in \text{Stab}_G^+(v^\omega)$. If $\pi(u') = 1$, then, since u', v commutes, $u' \in \mathcal{E}(\mathcal{M}) \neq \emptyset$. Otherwise, we have $\pi(u') \cdot v^n = \pi(u') \neq 1$ for all $n \geq 1$.

(iii) \Rightarrow (iv). If there is a non-elementary commuting pair $u \in (Q \setminus \{e\})^*, v \in \Sigma^*$, then by Lemma 5.2 the helix graph $\mathcal{H}_{|u|,|v|}$ contains the loop $(u, v) \longrightarrow (u, v)$, i.e., $\mathcal{H}_{|u|,|v|}$ is not singular.

(iv) \Rightarrow (i). If $\mathcal{E}(\mathcal{M}) \neq \emptyset$, then for any $u \in \mathcal{E}(\mathcal{M})$, by definition, there exists $v \in \Sigma^*$ satisfying

$$|u \cdot v^n|_e < |u|, \forall n \geq 1.$$

By a compactness argument, there exists some $m \geq 1$ with $|u \cdot v^m|_e = |u \cdot v^{m+k}|_e$ for every $k \geq 0$. Moreover, there exist indices $i > j \geq m$ satisfying $u \cdot v^j = u \cdot v^i$. As e acts like the identity, by erasing the (potential) occurrences of e in u , we obtain a word $u' \in (Q \setminus \{e\})^*$ such that u', v^{i-j} is a non-elementary pair of commuting words. Thus, we may assume $\mathcal{E}(\mathcal{M}) = \emptyset$. Now suppose $g \in \text{Stab}_{\langle \mathcal{M} \rangle}^+(\xi)$ for some $\xi \in \Sigma^\omega$ with $g \cdot \xi[:n] \neq 1$ for all $n \geq 1$. Hence, there are some word $u \in (Q \setminus \{e\})^*$ such that $\pi(u) = g$ and some indices $i > j \geq 1$ such that if we put $u' = u \cdot \xi[:j] \in (Q \setminus \{e\})^*$, then $u', \xi[j+1:i]$ is a non-elementary commuting pair. \square

Note that the previous proposition provides necessary conditions on $\mathcal{M} \in \mathcal{S}_a$ for tilings of $\overline{\mathcal{T}}(\mathcal{M})$ of the discrete plane to be aperiodic. As a result of Theorem 6.5 we immediately obtain the following undecidability result of checking the previous “continuity” condition.

Theorem 6.12. *Given an automaton $\mathcal{M} \in \mathcal{S}_a$, it is undecidable whether for any $\xi \in \Sigma^\omega$, $g \in \text{Stab}_G^+(\xi)$, there exists $n \geq 1$ with $g \cdot \xi[:n] = 1$ and $\mathcal{E}(\mathcal{M}) = \emptyset$.*

Moreover, by Proposition 6.11, taking the aperiodic 4-way deterministic tileset \mathcal{T} described in [21] and the associated automaton $\mathcal{M} \in \mathcal{S}_a$ with $\mathcal{T} = \overline{\mathcal{T}}(\mathcal{M})$ we get that \mathcal{M} actually satisfies the “continuity” conditions described in Proposition 6.11. The same paper raised the problem to determining the existence of an aperiodic reflection-closed tileset. A tileset \mathcal{T} with colored oriented edges is *closed under reflection* if for each tile in \mathcal{T} the reflection of this tile along a horizontal or vertical line also belongs to \mathcal{T} . Kari and Papasoglu considered the following tiling rule: a tiling of the plane, using tiles from a tileset \mathcal{T} which is closed under reflection, is said to be *valid* if two adjacent tiles meet along an edge with the same color and orientation and two tiles that are the reflection of each other are never adjacent. In the following, such a tiling will be called of *Kari-Papasoglu type*.

If we consider only the horizontal (vertical) symmetry we say that \mathcal{T} is *h*-reflection-closed (respectively, *v*-reflection-closed) tileset. Note that if \mathcal{T} is *h*-reflection-closed, then it is *ws*-deterministic (*es*-deterministic) if and only if it is *wn*-deterministic (respectively, *en*-deterministic). Similarly, if \mathcal{T} is *v*-reflection-closed, then it is *ws*-deterministic (*wn*-deterministic) if and only if it is *es*-deterministic (respectively, *en*-deterministic). Hence, if \mathcal{T} is reflection-closed and *xy*-deterministic for some $(x, y) \in \{(e, n), (e, s), (w, n), (w, s)\}$, then \mathcal{T} is necessarily 4-way deterministic. In [21] the authors raised the problem of finding a 4-way deterministic tileset which is valid, aperiodic and reflection-closed. Such a tileset would give an

example of a CAT(0) complex whose fundamental group is not hyperbolic and does not contain a subgroup isomorphic to \mathbb{Z}^2 , see [19, 21].

In this setting we can prove a statement analogous to Proposition 6.11. We will prove that the search for aperiodic h -reflection-closed tilesets that are ws - and wn -deterministic is related to the search for automata in the class \mathcal{S}_a whose set κ of singular points is empty and which are elementary-free, in the following sense.

First we need an analogous to Proposition 6.10 for h -reflection-closed tilings (resp. 4-way deterministic tilings). Let us define the *prefix-reduced relation*: we say that $u \in \tilde{Q}$ is smaller than $u' \in \tilde{Q}$ for the prefix-reduced relation if $u \leq_p u'$ holds and both u and u' are reduced. We define a property that will serve for the characterization of Kari-Papasoglu type tilings: an automaton $\mathcal{M} = (Q, \Sigma, \cdot, \circ) \in \mathcal{S}_a$ satisfies property **h-Maximal non Synchronizing** (resp. **4-way Maximal non Synchronizing**) if there exists an integer m such that, for all $v \in \Sigma^m$ (resp. reduced $v \in \widetilde{\Sigma^m}$), any reduced $u \in \tilde{Q}^*$ maximum in $\text{NSyn}_{\tilde{Q}}(v)$ for the prefix-reduced relation satisfies $uq \circ v \in \text{Syn}(\mathcal{M})$ for all $q \in \tilde{Q}$.

Proposition 6.13. *Let $\mathcal{M} = (Q, \Sigma, \cdot, \circ) \in \mathcal{S}_a$. Then $\overline{\mathcal{T}}$ admits an h -reflection-closed tiling (4-way deterministic tiling) if and only if property **h-Maximal non Synchronizing** (resp. **4-way Maximal non Synchronizing**) does not hold.*

Proof. Similar to the proof of Proposition 6.10, but avoiding patterns xx^{-1} that are not allowed in the Kari-Papasoglu type tilings. \square

A group generated by an automaton in the class \mathcal{S}_a is said to be *elementary-free* if the only relations that it contains are words whose restrictions become eventually all trivial, *i.e.* the set of its relations may be described as the set of words $u \in (\widetilde{Q \setminus \{e\}})^*$ with $\pi(u) = \mathbb{1}$, such that there exists an $n \geq 1$ for which $\overline{u \cdot v} = e^{k_v}$ for every $v \in \Sigma^{\geq n}$ and some integer k_v such that $|k_v| \leq |v|$.

In this context, we say that a helix graph $\tilde{\mathcal{H}}_{k,n}$ of an automaton $\mathcal{M} \in \mathcal{S}_a$ is *strongly-singular* whenever any cycle $(u, v) \longrightarrow (u', v') \longrightarrow \dots \longrightarrow (u, v)$ implies either $u \in \widetilde{\{e\}}^*$ or $\pi(u) = \mathbb{1}$. We have the following proposition analogous to Proposition 6.11.

Proposition 6.14. *Let $\mathcal{M} \in \mathcal{S}_a$. The following are equivalent:*

- (i) $\langle \mathcal{M} \rangle$ is elementary-free and the set κ of singular points is empty;
- (ii) $\langle \mathcal{M} \rangle$ is elementary-free and for any $v \in \Sigma^*$, $g \in \text{Stab}_{\langle \mathcal{M} \rangle}(v^\omega)$, there is $n \geq 1$ with $g \cdot v^\omega[:n] = \mathbb{1}$;
- (iii) for any $k, n \geq 1$, the helix graph $\tilde{\mathcal{H}}_{k,n}$ is strongly-singular;
- (iv) there is no non-elementary pair $u \in (\widetilde{Q \setminus \{e\}})^*$, $v \in \Sigma^*$ of commuting words.

Proof. (i) \Leftrightarrow (ii) follows from Lemma 4.2. Equivalence (ii) \Leftrightarrow (iii) may be proven in an analogous way as in Proposition 6.11. Equivalence (i) \Leftrightarrow (iv) is a consequence of Proposition 6.6. \square

Note that $\langle \mathcal{M} \rangle$ also acts naturally on $\tilde{\Sigma}^*$, in what follows we consider this action. Following [8], we say that a point $\xi \in \Sigma^\omega$ is *essentially non-trivial* when $\lim |\xi[:n]| \rightarrow_{n \rightarrow \infty} +\infty$. Moreover we say that a helix graph is *essentially-singular*

helix whenever, if $(u, v) \longrightarrow (u', v') \longrightarrow \cdots \longrightarrow (u, v)$ is a cycle, then v^ω is essentially non-trivial and either $u \in \widetilde{\{e\}}^*$, or $\pi(u) = 1$. We have the following proposition analogous to Proposition 6.11:

Proposition 6.15. *Let $\mathcal{M} \in \mathcal{S}_a$. The following are equivalent:*

- (i) $\langle \mathcal{M} \rangle$ is elementary-free and the set of essentially non-trivial singular points is empty;
- (ii) $\langle \mathcal{M} \rangle$ is elementary-free and for any $v \in \widetilde{\Sigma}^*$ such that v^ω is essentially non-trivial, $g \in \text{Stab}_{\langle \mathcal{M} \rangle}(v^\omega)$, there is $n \geq 1$ with $g \cdot v^\omega[:n] = 1$;
- (iii) for any $k, n \geq 1$, the helix graph $\widetilde{\mathcal{H}}_{k,n}$ is essentially singular;
- (iv) there is no non-elementary pair $u \in (\widetilde{Q \setminus \{e\}})^*$, $v \in \widetilde{\Sigma}^*$ of commuting words.

Proof. Similar to the proof of Prop. 6.14. □

Putting together all the previous results we may characterize aperiodic h -reflection-closed tilesets (resp. h -reflection-closed tilesets).

Theorem 6.16. *With the above notation. The following are equivalent:*

- (i) there is a ws - and wn -deterministic tileset \mathcal{T} which is h -reflection-closed (resp. a 4-way deterministic which is h - and v -reflection-closed) that tiles the discrete plane with aperiodic tilings of Kari-Papasoglu type;
- (ii) there is an automaton $\mathcal{M} \in \mathcal{S}_a$ such that $\langle \mathcal{M} \rangle$ is elementary-free, the set of singular points (resp. essentially non-trivial singular points) is empty, and, in the automaton \mathcal{B} obtained from $\mathcal{M} \sqcup \mathcal{M}^{-1}$ identifying the two sinks e and e^{-1} , property **h -Maximal non Synchronizing** (resp. **4-way Maximal non Synchronizing**) does not hold.

Proof. If \mathcal{T} is h -reflection-closed, we may fix a direction and divide the colors of the vertical edges of \mathcal{T} into two distinct and disjoint sets $Q \sqcup Q^{-1}$, while we put for Σ the set of colors of the horizontal edges of the tiles in \mathcal{T} . Note that, for each tile (q, a, p, b) , $q, p \in \widetilde{Q}$, $a, b \in \Sigma$, the corresponding horizontally reflected tile is (q^{-1}, a, p^{-1}, b) . The partition $Q \sqcup Q^{-1}$ induces a partition $\mathcal{T}^+ \sqcup \mathcal{T}^-$ on \mathcal{T} in the obvious way. Consider an associated automaton $\mathcal{M} \in \mathcal{S}_a$ such that $\overline{\mathcal{T}}(\mathcal{M}) = \mathcal{T}^+$ (as in Section 5). Note that $\overline{\mathcal{T}}(\mathcal{M} \sqcup \mathcal{M}^{-1}) = \mathcal{T}$.

Conversely, to any automaton $\mathcal{M} \in \mathcal{S}_a$, the tileset $\overline{\mathcal{T}}(\mathcal{M} \sqcup \mathcal{M}^{-1})$ is ws - and wn -deterministic, and it is h -reflection-closed. By an argument very similar to the proof of Propositions 6.2 and 6.4, it is not difficult to see that, in the previous correspondence, \mathcal{M} has a non-elementary pair $u \in (\widetilde{Q \setminus \{e\}})^*$, $v \in \Sigma^*$ of commuting words where u is non-trivial and reduced if and only if the corresponding tileset $\overline{\mathcal{T}}(\mathcal{M} \sqcup \mathcal{M}^{-1})$ admits a periodic tiling (in the sense of Kari-Papasoglu). Hence, the equivalence in the statement follows from Proposition 6.13 for the existence of a tiling of Kari-Papasoglu type and Proposition 6.14 for its aperiodicity. The proof for the 4-way case is similar and uses Proposition 6.15. □

The last theorem gives a characterization of specific Wang tilings in the language of Mealy automata. This is another motivation to further explore this connection.

7. SOME OPEN PROBLEMS

Problem 1. Let $\mathcal{M} \in \mathcal{S}_a$ be an automaton generating a free group. Is it always the case that there is a point in the boundary whose Schreier graph is infinite?

Problem 2. Given that \mathcal{M} is minimized, can two singular points have isomorphic infinite Schreier Graphs?

Problem 3. Given a Mealy automaton \mathcal{M} , is it decidable whether $\mathcal{P}(\mathcal{M}) = \emptyset$?

Problem 4. Are there interesting classes of automata where the non-elementary commuting pair is decidable?

REFERENCES

- [1] A. Akhavi, I. Klimann, S. Lombardy, J. Mairesse, and M. Picantin. On the finiteness problem for automaton (semi)groups. *International Journal of Algebra and Computation*, 22(6):1–26, 2012.
- [2] L. Bartholdi and V. Nekrashevych. Thurston equivalence of topological polynomials. *Acta Mathematica*, 197(1):1–51, 2006.
- [3] L. Bartholdi and P. V. Silva. Groups defined by automata. In J.-É. Pin, editor, *AutoMathA Handbook*. European Mathematical Society, to appear. <http://arxiv.org/abs/1012.1531>.
- [4] R. Berger. The undecidability of the domino problem. *Memoirs of the American Mathematical Society*, 66:72, 1966.
- [5] I. Bondarenko. private communication.
- [6] I. Bondarenko, D. D'Angeli, and T. Nagnibeda. Ends of schreier graphs and cut-points of limit spaces of self-similar groups. <http://arxiv.org/abs/1601.07587>.
- [7] I. Bondarenko, D. D'Angeli, and E. Rodaro. The lamplighter group $z_3 \wr z$ generated by a bireversible automaton. *Communications in Algebra*, to appear. <http://arxiv.org/abs/1502.07981>.
- [8] D. D'Angeli and E. Rodaro. Freeness of automaton groups vs boundary dynamics. <http://arxiv.org/abs/1502.07981>.
- [9] D. D'Angeli and E. Rodaro. Groups and semigroups defined by colorings of synchronizing automata. *International Journal of Algebra and Computation*, 24(6):773–793, 2014.
- [10] D. D'Angeli and E. Rodaro. A geometric approach to (semi)-groups defined by automata via dual transducers. *Geometriae Dedicata*, 174:375–400, 2015.
- [11] N. Davis, M. Elder, and L. Reeves. Non-contracting groups generated by $(3, 2)$ -automata. *Algebra and Discrete Mathematics*, 17(1):20–32, 2014.
- [12] P. Gillibert. The finiteness problem for automaton semigroups is undecidable. *International Journal of Algebra and Computation*, 24-1:1–9, 2014.
- [13] Y. Glasner and S. Mozes. Automata and square complexes. *Geometriae Dedicata*, 111:43–64, 2005.
- [14] Th. Godin, I. Klimann, and M. Picantin. On torsion-free semigroups generated by invertible reversible Mealy automata. In *Language and automata theory and applications*, volume 8977 of *Lecture Notes in Comput. Sci.*, pages 328–339. Springer, Cham, 2015.
- [15] R. I. Grigorchuk. Some topics of the dynamics of group actions on rooted trees. *The Proceedings of the Steklov Institute of Math.*, 273:1–118, 2011.
- [16] R. I. Grigorchuk and D. Savchuk. Self-similar groups acting essentially freely on the boundary of the binary rooted tree. In *Group theory, combinatorics, and computing*, volume 611 of *Contemp. Math.*, pages 9–48. Amer. Math. Soc., Providence, RI, 2014.
- [17] R. I. Grigorchuk and Z. Šuník. Asymptotic aspects of Schreier graphs and Hanoi Towers groups. *Comptes Rendus Mathématique. Académie des Sciences. Paris*, 342(8):545–550, 2006.
- [18] R. I. Grigorchuk and A. Żuk. On a torsion-free weakly branch group defined by a three state automaton. *International Journal of Algebra and Computation*, 12(1-2):223–246, 2002. International Conference on Geometric and Combinatorial Methods in Group Theory and Semigroup Theory (Lincoln, NE, 2000).
- [19] M. Gromov. Hyperbolic manifolds, groups and actions. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook)*.

- Brook, N.Y., 1978*), volume 97 of *Ann. of Math. Stud.*, pages 183–213. Princeton Univ. Press, Princeton, N.J., 1981.
- [20] M. Kambites, P. V. Silva, and B. Steinberg. The spectra of lamplighter groups and Cayley machines. *Geometriae Dedicata*, 120:193–227, 2006.
 - [21] J. Kari and P. Papasoglu. Deterministic aperiodic tile sets. *Geometric and Functional Analysis*, 9(2):353–369, 1999.
 - [22] I. Klimann, M. Picantin, and D. Savchuk. A connected 3-state reversible mealy automaton cannot generate an infinite burnside group. In *Developments in Language Theory - 19th International Conference, DLT 2015, Liverpool, UK, July 27-30, 2015, Proceedings.*, pages 313–325, 2015.
 - [23] B. Le Gloannec. The 4-way deterministic periodic domino problem is undecidable. <https://hal.archives-ouvertes.fr/hal-00985482>, 2014.
 - [24] O. Macedońska, V. Nekrashevych, and V. Sushchansky. Commensurators of groups and reversible automata. *Dopovidī Natsional'noi Akademii Nauk Ukraini. Matematika. Prirodosnavstvo. Tekhnichni Nauki*, 12:36–39, 2000.
 - [25] V. Nekrashevych. *Self-similar groups*, volume 117 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005.
 - [26] V. Nekrashevych. Free subgroups in groups acting on rooted trees. *Groups, Geometry, and Dynamics*, 4(4):847–862, 2010.
 - [27] D. Perrin and J.-É. Pin. *Infinite words : automata, semigroups, logic and games*. Pure and applied mathematics. Academic, London, San Diego (Calif.), 2004.
 - [28] R. M. Robinson. Undecidability and nonperiodicity for tilings of the plane. *Invent. Math.*, 12:177–209, 1971.
 - [29] D. Savchuk and S. Sidki. Affine automorphisms of rooted trees. *Geometriae Dedicata*, to appear. <http://arxiv.org/abs/1502.07981>.
 - [30] D. Savchuk and Y. Vorobets. Automata generating free products of groups of order 2. *Journal of Algebra*, 336:53–66, 2011.
 - [31] S. Sidki. Automorphisms of one-rooted trees: growth, circuit structure, and acyclicity. *Journal of Mathematical Sciences (New York)*, 100(1):1925–1943, 2000. Algebra, 12.
 - [32] B. Steinberg, M. Vorobets, and Y. Vorobets. Automata over a binary alphabet generating free groups of even rank. *International Journal of Algebra and Computation*, 21(1-2):329–354, 2011.
 - [33] W. Thomas. Automata on infinite objects. In *Handbook of theoretical computer science, Vol. B*, pages 133–191. Elsevier, Amsterdam, 1990.
 - [34] M. Vorobets and Y. Vorobets. On a free group of transformations defined by an automaton. *Geometriae Dedicata*, 124:237–249, 2007.
 - [35] M. Vorobets and Y. Vorobets. On a series of finite automata defining free transformation groups. *Groups, Geometry, and Dynamics*, 4(2):377–405, 2010.
 - [36] Y. Vorobets. Notes on the Schreier graphs of the Grigorchuk group. In *Dynamical systems and group actions*, volume 567 of *Contemp. Math.*, pages 221–248. Amer. Math. Soc., Providence, RI, 2012.
 - [37] E. I. Zel'manov. Solution of the restricted Burnside problem for groups of odd exponent. *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya*, 54-1:42–59, 221, 1990.
 - [38] E. I. Zel'manov. Solution of the restricted Burnside problem for 2-groups. *Matematicheskii Sbornik*, 182-4:568–592, 1991.

INSTITUT FÜR DISKRETE MATHEMATIK, TECHNISCHE UNIVERSITÄT GRAZ, STEYRERGASSE 30,
8010 GRAZ, AUSTRIA.

E-mail address: `dangeli@math.tugraz.at`

IRIF, UMR 8243 UNIVERSITÉ PARIS DIDEROT & CNRS, BÂTIMENT SOPHIE GERMAIN, 75205
PARIS CEDEX 13, FRANCE.

E-mail address: `godin@irif.fr`

IRIF, UMR 8243 UNIVERSITÉ PARIS DIDEROT & CNRS, BÂTIMENT SOPHIE GERMAIN, 75205
PARIS CEDEX 13, FRANCE.

E-mail address: `klimann@irif.fr`

IRIF, UMR 8243 UNIVERSITÉ PARIS DIDEROT & CNRS, BÂTIMENT SOPHIE GERMAIN, 75205
PARIS CEDEX 13, FRANCE.

E-mail address: `picantin@irif.fr`

DEPARTMENT OF MATHEMATICS, POLITECNICO DI MILANO, PIAZZA LEONARDO DA VINCI, 32,
20133 MILANO, ITALY

E-mail address: `emanuele.rodaro@polimi.it`